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SOLUTIONS MANUAL TO ACCOMPANY ROMER

ADVANCED MACROECONOMICS

JEFFREY ROHALY

University of California, Berkeley



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Solutions Manual to Accompany Romer
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Preface

This solutions manual is designed to accompany Advanced Macroeconomics by David Romer. It contains suggested solutions to all of the end-of-chapter problems, with the exception of some of the empirical exercises. I have made every attempt to provide a very detailed, step-by-step answer to each problem. Only basic algebraic manipulations have been occasionally omitted.

Comments and suggestions for improvement are welcomed and should be addressed to Jeffrey Rohaly, Department of Economics, University of California, Berkeley, California 94720. I can also be reached via e-mail at rohaly@econ.berkeley.edu.

I would like to thank David Romer for his generous help with this manual. He provided many suggestions and corrections that have greatly improved the quality of these answers. Any errors that may remain are, of course, my own.

John Doe

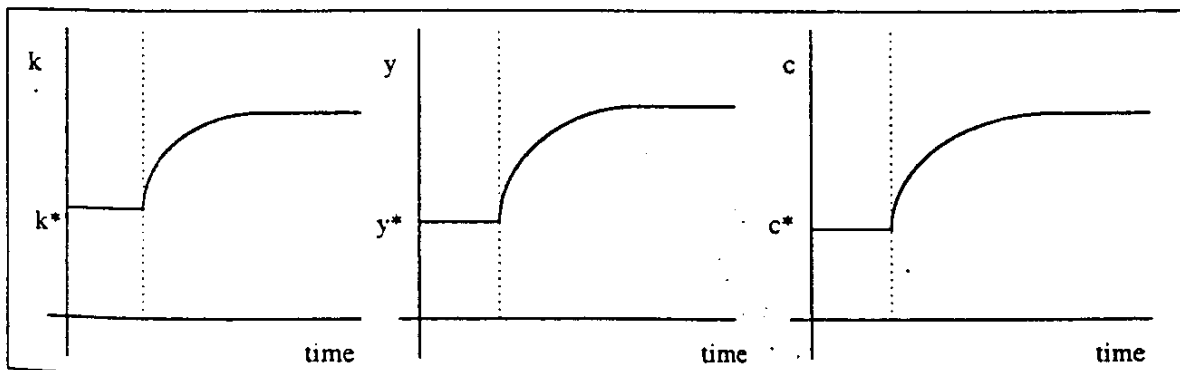
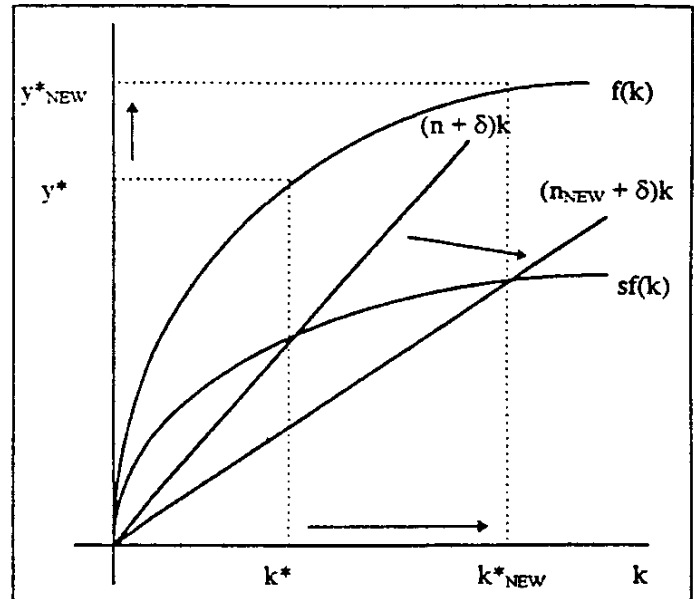
SOLUTIONS TO CHAPTER 1

Problem 1.1

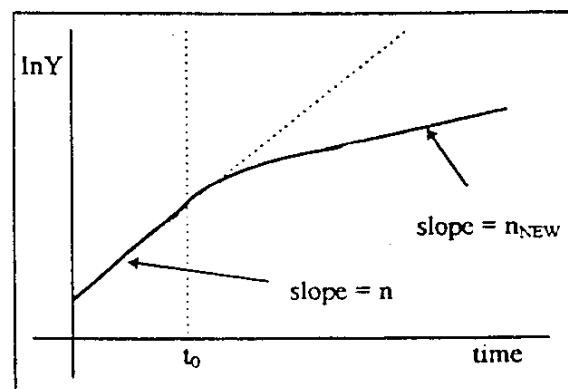
a) Since there is no technological progress, we can carry out the entire analysis in terms of capital and output per worker rather than capital and output per unit of effective labor. With A constant, they behave the same. Thus for the purposes of this problem we can define $y \equiv Y/L$ and $k \equiv K/L$.

The fall in the population growth rate makes the break-even investment line flatter. In the absence of technological progress, the per unit time change in k , capital per worker, is given by $\dot{k} = sf(k) - (\delta + n)k$. Since $\dot{k}$ was 0 before the decrease in n — the economy was on a balanced growth path — the decrease in n causes $\dot{k}$ to become positive. At k^* , actual investment per worker, $sf(k^*)$, now exceeds break-even investment per worker, $(n_{\text{NEW}} + \delta)k^*$. Thus k moves to a new higher balanced-growth-path level — see the diagram.

As k rises, y — output per worker — also rises. Since a constant fraction of output is saved, c — consumption per worker — rises as y rises. This is all summarized in the diagrams below.



b) By definition, output can be written as $Y \equiv Ly$. Thus the growth rate of output is $\dot{Y}/Y = \dot{L}/L + \dot{y}/y$. On the initial balanced growth path, $\dot{y}/y = 0$ — output per worker is constant — so $\dot{Y}/Y = \dot{L}/L = n$. On the final balanced growth path, $\dot{y}/y = 0$ again — output per worker is constant again — and so $\dot{Y}/Y = \dot{L}/L = n_{\text{NEW}} < n$. In the end, output will be growing at a permanently lower rate.



What happens during the transition? Look at the production function $Y = F(K, AL)$. On the initial balanced growth path AL , K and thus Y are all growing at rate n . Then suddenly AL begins growing at some new lower rate n_{NEW} . Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and of AL (which is growing at n_{NEW}). Thus during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally, in the end, K , AL and thus Y are all growing at the new lower n_{NEW} .

Problem 1.2

a) The equation describing the evolution of the capital stock per unit of effective labor is given by :

$$(1) \quad \dot{k} = sf(k) - (n + g + \delta)k$$

Substituting in for the intensive form of the Cobb-Douglas — $f(k) = k^\alpha$ — yields:

$$\dot{k} = sk^\alpha - (n + g + \delta)k$$

On the balanced growth path, $\dot{k}$ is zero — investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k^* , we have $sk^{*\alpha} = (n + g + \delta)k^*$. Rearranging to solve for k^* yields:

$$(2) \quad k^* = [s/(n + g + \delta)]^{1/(1-\alpha)}$$

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (2) into the intensive form of the production function — $y = k^\alpha$:

$$(3) \quad y^* = [s/(n + g + \delta)]^{\alpha/(1-\alpha)}$$

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1 - s)y^*$.

Substituting equation (3) into this expression yields:

$$(4) \quad c^* = (1 - s)[s/(n + g + \delta)]^{\alpha/(1-\alpha)}$$

b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k , take equation (2) — the balanced-growth-path level of k — and rearrange it to solve for s :

$$(5) \quad s = (n + g + \delta)k^{*1-\alpha}$$

Now substitute equation (5) into equation (4):

$$c^* = [1 - (n + g + \delta)k^{*1-\alpha}] \cdot [(n + g + \delta)k^{*1-\alpha} / (n + g + \delta)]^{\alpha/(1-\alpha)}$$

After some straightforward algebraic manipulation, this simplifies to:

$$(6) \quad c^* = k^{*\alpha} - (n + g + \delta)k^*$$

Equation (6) can be easily interpreted. Consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor, which on the balanced growth path is the same as break-even investment per unit of effective labor, $(n + g + \delta)k^*$.

Now use equation (6) to maximize c^* with respect to k^* . The first-order condition is given by :

$$\partial c^* / \partial k^* = \alpha k^{*\alpha-1} - (n + g + \delta) = 0$$

or simply:

$$(7) \quad \alpha k^{*\alpha-1} = (n + g + \delta)$$

Note that equation (7) is just a specific form of $f'(k^*) = (n + g + \delta)$, which is the general condition that implicitly defines the golden-rule level of capital per unit of effective labor. It has a graphical interpretation as the level of k where the slope of the intensive form of the production function is equal to the slope of the break-even investment line.

Solving equation (7) for the golden-rule level of k yields:

$$(8) \quad k^*_{GR} = [\alpha / (n + g + \delta)]^{1/(1-\alpha)}$$

c) To get the saving rate that will yield the golden-rule level of k , substitute equation (8) into (5):

$$s_{GR} = (n + g + \delta) \cdot [\alpha / (n + g + \delta)]^{(1-\alpha)/(1-\alpha)}$$

which simplifies to:

$$(9) \quad s_{GR} = \alpha$$

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

Problem 1.3

a) Multiply the amounts of capital and effective labor by a non-negative constant c :

$$F(cK, cAL) = [(cK)^{(\sigma-1)/\sigma} + (cAL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} = [c^{(\sigma-1)/\sigma} (K^{(\sigma-1)/\sigma} + (AL)^{(\sigma-1)/\sigma})]^{\sigma/(\sigma-1)}$$

and simplifying further yields:

$$F(cK, cAL) = c [K^{(\sigma-1)/\sigma} + (AL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} = cF(K, AL)$$

Thus the CES production function does exhibit constant returns to scale.

b) Divide the production function through by AL :

$$Y/AL = [(1/AL)^{(\sigma-1)/\sigma} (K^{(\sigma-1)/\sigma} + (AL)^{(\sigma-1)/\sigma})]^{\sigma/(\sigma-1)} = [(K/AL)^{(\sigma-1)/\sigma} + (AL/AL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)}$$

Define $k \equiv K/AL$ and $y \equiv Y/AL \equiv f(k)$ and thus the production function in intensive form is given by:

$$(1) \quad f(k) = [k^{(\sigma-1)/\sigma} + 1]^{\sigma/(\sigma-1)}$$

c) Use equation (1) to take the derivative of $f(k)$ with respect to k :

$$f'(k) = [\sigma/(\sigma-1)] \cdot [k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \cdot [(\sigma-1)/\sigma] \cdot k^{-1/\sigma}$$

or simply:

$$(2) \quad f'(k) = [k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \cdot k^{-1/\sigma}$$

With $k > 0$, $f'(k)$ will be positive.

Now use equation (2) to find the second derivative of $f(k)$ with respect to k :

$$f''(k) = \left\{ [1/(\sigma-1)] \cdot [k^{(\sigma-1)/\sigma} + 1]^{(2-\sigma)/(\sigma-1)} \cdot [(\sigma-1)/\sigma] \cdot k^{-1/\sigma} \right\} \cdot k^{-1/\sigma} + [k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \cdot [(-1/\sigma)k^{(-1-\sigma)/\sigma}]$$

We can factor the expression as follows:

$$f''(k) = (1/\sigma)k^{-1/\sigma} [k^{(\sigma-1)/\sigma} + 1]^{1/(\sigma-1)} \cdot \left\{ k^{-1/\sigma} [k^{(\sigma-1)/\sigma} + 1]^{-1} - k^{-1} \right\}$$

and then using equation (2), this can be written as:

$$(3) \quad f''(k) = (1/\sigma)f'(k) \cdot \left\{ k^{-1/\sigma} [k^{(\sigma-1)/\sigma} + 1]^{-1} - k^{-1} \right\}$$

We know that for $k > 0$, $(1/\sigma)f'(k) > 0$. Thus $f''(k)$ will be negative when the following condition holds

$$k^{-1/\sigma} [k^{(\sigma-1)/\sigma} + 1]^{-1} - k^{-1} < 0 \quad \Rightarrow \quad k^{-1/\sigma} - k^{-1/\sigma} - k^{-1} < 0$$

or simply when:

$$-k^{-1} < 0 \Rightarrow 1/k > 0$$

Thus $f''(k) < 0$ for all positive values of k .

d) We have:

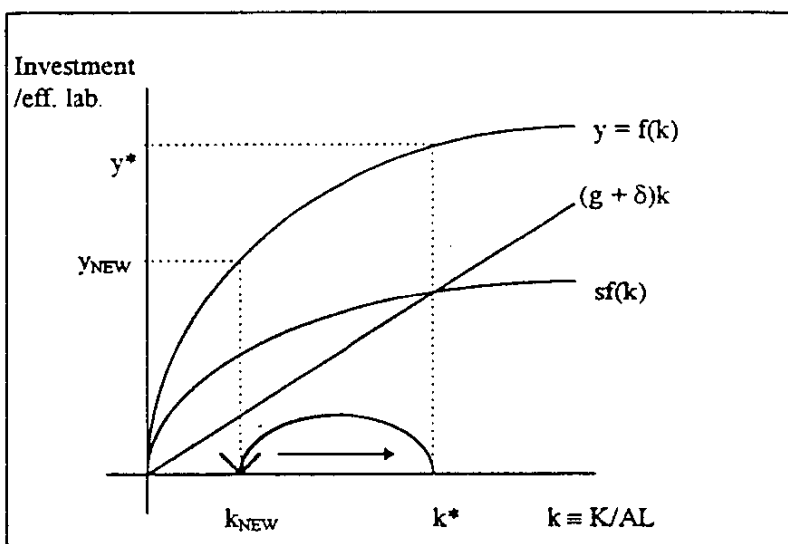
$$\lim_{k \rightarrow 0} f'(k) = \lim_{k \rightarrow 0} \left[k^{(\sigma-1)/\sigma} + 1 \right]^{1/(\sigma-1)} \left(\frac{1}{k} \right)^{1/\sigma} = \infty \quad \text{with } \sigma > 1$$

and:

$$\lim_{k \rightarrow \infty} f'(k) = \lim_{k \rightarrow \infty} \left[k^{(\sigma-1)/\sigma} + 1 \right]^{1/(\sigma-1)} \left(\frac{1}{k} \right)^{1/\sigma} = \lim_{k \rightarrow \infty} \left[\left(\frac{1}{k} \right)^{(1-\sigma)/\sigma} + 1 \right]^{1/(\sigma-1)} \left(\frac{1}{k} \right)^{1/\sigma} = 0 \quad \text{with } \sigma < 1$$

Problem 1.4

a) At some time — call it t_0 — there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k^* to k_{NEW} . We can see this by simply looking at the definition — $k \equiv K/AL$ — and we have had an increase in L without a jump in K or A . Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the diagram, it falls from y^* to y_{NEW} .



b) Now at this lower k_{NEW} , actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{\text{NEW}}) > (g + \delta)k_{\text{NEW}}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW} . Thus k begins rising back toward k^* . As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y^* .

c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k^* . At k^* , investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k^* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.

Problem 1.5

The derivative of $y^* = f(k^*)$ with respect to n is given by:

$$(1) \quad \partial y^* / \partial n = f'(k^*) [\partial k^* / \partial n]$$

To find $\partial k^* / \partial n$, use the equation for the evolution of the capital stock per unit of effective labor,

$\dot{k} = sf(k) - (n + g + \delta)k$. In addition, use the fact that on a balanced growth path, $\dot{k} = 0$, $k = k^*$ and thus $sf(k^*) = (n + g + \delta)k^*$. Taking the derivative of both sides of this expression with respect to n yields:

$$sf'(k^*) \cdot \frac{\partial k^*}{\partial n} = (n + g + \delta) \cdot \frac{\partial k^*}{\partial n} + k^*$$

and rearranging yields:

$$(2) \frac{\partial k^*}{\partial n} = \frac{k^*}{sf'(k^*) - (n + g + \delta)}$$

Substituting equation (2) into equation (1) gives us:

$$(3) \frac{\partial y^*}{\partial n} = f'(k^*) \cdot \left[\frac{k^*}{sf'(k^*) - (n + g + \delta)} \right]$$

Rearrange the condition that implicitly defines $k^* - sf(k^*) = (n + g + \delta)k^*$ and solve for s yielding:

$$(4) s = (n + g + \delta)k^*/f(k^*)$$

Substitute equation (4) into equation (3):

$$(5) \frac{\partial y^*}{\partial n} = \frac{f'(k^*)k^*}{[(n + g + \delta)f'(k^*)k^*/f(k^*)] - (n + g + \delta)}$$

To turn this into the elasticity that we want, multiply both sides of equation (5) by n/y^* :

$$\frac{n}{y^*} \cdot \frac{\partial y^*}{\partial n} = \frac{n}{(n + g + \delta)} \cdot \frac{f'(k^*)k^*/f(k^*)}{[f'(k^*)k^*/f(k^*)] - 1}$$

Using the definition that $\alpha_K(k^*) \equiv f'(k^*)k^*/f(k^*)$ gives us:

$$(6) \frac{n}{y^*} \cdot \frac{\partial y^*}{\partial n} = - \frac{n}{(n + g + \delta)} \cdot \left[\frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)} \right]$$

Now, with $\alpha_K(k^*) = 1/3$, $g = 2\%$ and $\delta = 3\%$, we need to calculate the effect on y^* of a fall in n from 2% to 1%. Using the midpoint of $n = 0.015$ to calculate the elasticity gives us:

$$\frac{n}{y^*} \cdot \frac{\partial y^*}{\partial n} = - \frac{0.015}{(0.015 + 0.02 + 0.03)} \cdot \left(\frac{1/3}{1 - 1/3} \right) \cong -0.12$$

So this 50% drop in the population growth rate (from 2% to 1%) will lead to a $(-0.50)(-0.12) \cong 0.06$ or 6% increase in the level of output per unit of effective labor. This illustrates the point that observed differences in population growth rates across countries are not nearly enough to account for differences in y that we see.

Problem 1.6

a) Assuming that investment rises by the full amount of the fall in the deficit, the share of output that is devoted to investment – the saving rate – should rise from $s = 0.15$ to $s_{\text{NEW}} = 0.18$. Note that this is a 20% increase in the saving rate (0.03 is 20% of 0.15). From equation (1.22) in the text, the elasticity of output with respect to the saving rate is:

$$(1) \frac{s}{y^*} \cdot \frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)}$$

where $\alpha_K(k^*)$ is the share of income paid to capital (assuming that capital is paid its marginal product).

We are told to assume that $\alpha_K(k^*) = 1/3$. Substituting this in gives us:

$$\frac{s}{y^*} \cdot \frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1 - \alpha_K(k^*)} = \frac{1/3}{1 - 1/3} = \frac{1}{2}$$

Thus the elasticity of output with respect to the saving rate is $1/2$. So this 20% increase in the saving rate – from $s = 0.15$ to $s_{\text{NEW}} = 0.18$ – will cause output to rise relative to what it would have been by about 10%. For such a huge policy change – total elimination of the budget deficit – this appears to be a rather modest benefit. [Note that the analysis has really been carried out in terms of output per unit of effective

labor. Since the paths of A and L are not affected, however, if output per unit of effective labor rises by 10%, output itself is also 10% higher than what it would have been.]

b) Consumption will rise even less than output. Although output winds up 10% higher than what it would have been, the fact that the saving rate is higher means that we are consuming a smaller fraction of output than before the deficit reduction. Thus we do not get to enjoy the entire 10% increase in output as increased consumption. Note that by consumption, we are implicitly including government purchases. We can calculate the elasticity of consumption with respect to the saving rate. On the balanced growth path, consumption is given by:

$$(2) \quad c^* = (1 - s)y^*$$

Taking the derivative with respect to s yields:

$$(3) \quad \frac{\partial c^*}{\partial s} = -y^* + (1 - s) \cdot \frac{\partial y^*}{\partial s}$$

To turn this into an elasticity, multiply both sides of equation (3) by s/c^* :

$$\frac{\partial c^*}{\partial s} \cdot \frac{s}{c^*} = \frac{-y^* s}{(1 - s)y^*} + (1 - s) \cdot \frac{\partial y^*}{\partial s} \cdot \frac{s}{(1 - s)y^*}$$

where we have substituted $c^* = (1 - s)y^*$ on the right-hand side. Simplifying gives us:

$$(4) \quad \frac{\partial c^*}{\partial s} \cdot \frac{s}{c^*} = \frac{-s}{(1 - s)} + \frac{\partial y^*}{\partial s} \cdot \frac{s}{(1 - s)y^*}$$

From part a, the second term -- the elasticity of output with respect to the saving rate -- is equal to $1/2$. We can use the midpoint between $s = 0.15$ and $s_{\text{NEW}} = 0.18$ to calculate the elasticity:

$$\frac{\partial c^*}{\partial s} \cdot \frac{s}{c^*} = \frac{-0.165}{(1 - 0.165)} + 0.5 \approx 0.30$$

Thus the elasticity of consumption with respect to the saving rate is approximately 0.3. So this 20% increase in the saving rate -- from $s = 0.15$ to $s_{\text{NEW}} = 0.18$ -- will lead to consumption being approximately 6% above what it would have been. Again, the Solow model yields only modest benefits from a rather large change in government policy.

c) The immediate effect of the deficit reduction is that consumption falls. Although y^* does not jump immediately -- it only begins to move towards its new, higher balanced-growth-path level -- we are now saving a greater fraction and thus consuming a smaller fraction of this same y^* . At the moment of the rise in s by 3 percentage points -- since $c = (1 - s)y^*$ and y^* is unchanged -- c falls. In fact, the percentage change in c will be the percentage change in $(1 - s)$. Now, $(1 - s)$ falls from 0.85 to 0.82, which is approximately a 3.5% drop -- 0.03 is about 3.5% of 0.85. Thus at the moment of the rise in s , consumption falls by about three and a half percent.

We can use some results from the text on the speed of convergence to determine how long it takes for consumption to return to what it would have been without the deficit reduction. After the initial rise in s , s remains constant throughout. Since $c = (1 - s)y$, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In the text it is shown that the rate of convergence of k and y -- after a linear approximation -- is given by $\lambda = (1 - \alpha_K)(n + g + \delta)$. With $(n + g + \delta) = 6\%$ per year and $\alpha_K = 1/3$, this yields $\lambda \approx 4\%$. This means that k and y move 4% of the remaining distance toward their balanced-growth-path values of k^* and y^* each year. Since c is proportional to y -- $c = (1 - s)y$ -- it also approaches its new balanced-growth-path value at that same constant rate. That is, analogous to equation (1.26) in the text, we could write:

$$(5) \quad c(t) - c^* \approx e^{-(1 - \alpha_K)(n + g + \delta)t} [c(0) - c^*]$$

or equivalently:

$$(6) e^{-\lambda t} = \frac{c(t) - c^*}{c(0) - c^*}$$

The term on the right-hand side of equation (6) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 3.5% and will eventually be 6% higher than it would have been. Thus it must change by 9.5% on the way to the balanced growth path. It will therefore be equal to what it would have been, $3.5\%/9.5\% \cong 36.8\%$ of the way to the new balanced growth path. Equivalently, this is when the remaining distance to the new balanced growth path is 63.2% of the original distance. In order to find out how long this will take, we need to find a t^* that solves:

$$(7) e^{-\lambda t^*} = 0.632$$

Taking the natural log of both sides of equation (7) yields:

$$-\lambda \cdot t^* = \ln(0.632) \Rightarrow t^* = 0.459/0.04$$

and thus:

$$(8) t^* \cong 11.5 \text{ years}$$

It will take a fairly long time -- over a decade -- for consumption to return to what it would have been in the absence of the deficit reduction.

Problem 1.7

a) Define the marginal product of labor as $w \equiv \partial F(K, AL)/\partial L$. Then write the production function as $Y = ALf(k) = ALf(K/AL)$. Taking the partial derivative of output with respect to L yields:

$$(1) w \equiv \partial Y/\partial L = ALf'(k)[-K/AL^2] + Af(k) = A[(-K/AL)f'(k) + f(k)] = A[f(k) - kf'(k)]$$

So we do have $w = A[f(k) - kf'(k)]$.

b) Define the marginal product of capital as $r \equiv \partial F(K, AL)/\partial K$. Again, writing the production function as $Y = ALf(k) = ALf(K/AL)$ and now taking the partial derivative of output with respect to K yields:

$$(2) r \equiv \partial Y/\partial K = ALf'(k)[1/AL] = f'(k)$$

Substitute equations (1) and (2) into $wL + rK$:

$$wL + rK = A[f(k) - kf'(k)]L + f'(k)K = ALf(k) - f'(k)[K/AL]AL + f'(k)K$$

Simplifying gives us:

$$(3) wL + rK = ALf(k) - f'(k)K + f'(k)K = ALf(k) \equiv ALF(K/AL, 1)$$

Finally, since F is constant returns to scale, equation (3) can be rewritten as:

$$(4) wL + rK = F(ALK/AL, AL) = F(K, AL)$$

c) As shown above, $r = f'(k)$. Since k is constant on a balanced growth path, so is $f'(k)$ and thus so is r . In other words, on a balanced growth path, $\dot{r}/r = 0$. This fits one of Kaldor's stylized facts about growth which is that the return to capital is approximately constant. Since capital is paid its marginal product, the share of output going to capital is rK/Y . On a balanced growth path:

$$(5) \frac{\dot{(rK/Y)}}{(rK/Y)} = \dot{r}/r + \dot{K}/K - \dot{Y}/Y = 0 + (n + g) - (n + g) = 0$$

Thus on a balanced growth path, the share of output going to capital is constant. Since the shares of output going to capital and labor sum to 1, this implies that the share of output going to labor is also constant on the balanced growth path.

d) We need to determine what is happening to the growth rate of w as k rises toward k^* . As shown above, $w = A[f(k) - kf'(k)]$. Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

$$(6) \quad \frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \frac{[f(k) - kf'(k)]}{[f(k) - kf'(k)]} = g + \frac{[f'(k)\dot{k} - kf''(k)\dot{k}]}{f(k) - kf'(k)} = g + \frac{-kf''(k)\dot{k}}{f(k) - kf'(k)} > g$$

This is true because the denominator is positive since $f(k)$ is a concave function, while the numerator is positive because k and $\dot{k}$ are positive while $f''(k)$ is negative. Thus the marginal product of labor is growing faster than on the balanced growth path. Intuitively, the marginal product of labor rises by the rate of growth of the effectiveness of labor on the balanced growth path since that is what is increasing the marginal product of labor. As we move from k to k^* , however the amount of capital per unit of effective labor is also rising which also makes labor more productive and this increases the marginal product of labor even more.

The growth rate of the marginal product of capital, r , is:

$$(7) \quad \frac{\dot{r}}{r} = \frac{[f''(k)]}{f'(k)} = \frac{f''(k)\dot{k}}{f'(k)}$$

This growth rate is negative since $f'(k) > 0$, $f''(k) < 0$ and $\dot{k} > 0$ as k rises towards k^* . Thus as the economy moves from k to k^* , the marginal product of capital falls. That is, it grows at a rate less than on the balanced growth path where its growth rate is 0.

Problem 1.8

a) By definition a balanced growth path is when all the variables of the model are growing at constant rates. Despite the differences between this model and the usual Solow model, it turns out that we can again show that the economy will converge to a balanced growth path by examining the behavior of $k \equiv K/AL$. Taking the time derivative of both sides of the definition of $k \equiv K/AL$ gives us:

$$(1) \quad \dot{k} = \left(\frac{\dot{K}}{AL} \right) = \frac{\dot{K}(AL) - K[\dot{L}A - \dot{A}L]}{(AL)^2} = \frac{\dot{K}}{AL} - \frac{K}{AL} \left[\frac{\dot{L}A + \dot{A}L}{AL} \right] = \frac{\dot{K}}{AL} - k \left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A} \right)$$

Substituting the capital accumulation equation — $\dot{K} = [\partial F(K, AL)/\partial K]K - \delta K$ — and the constant growth rates of the labor force and technology — $\dot{L}/L = n$ and $\dot{A}/A = g$ — into equation (1) yields:

$$(2) \quad \dot{k} = \frac{[\partial F(K, AL)/\partial K]K - \delta K}{AL} - (n + g)k = \frac{\partial F(K, AL)}{\partial K} \cdot k - \delta k - (n + g)k$$

Substituting $\partial F(K, AL)/\partial K = f'(k)$ into equation (2) gives us $\dot{k} = f'(k)k - \delta k - (n + g)k$ or simply:

$$(3) \quad \dot{k} = [f'(k) - (n + g + \delta)]k$$

Capital per unit of effective labor will be constant when $\dot{k} = 0$, i.e. when $[f'(k) - (n + g + \delta)]k = 0$. This condition will hold if $k = 0$ (a case we will ignore) or when $f'(k) - (n + g + \delta) = 0$. Thus the balanced-growth-path level of the capital stock per unit of effective labor is implicitly defined by $f'(k^*) = (n + g + \delta)$. Since capital per unit of effective labor, $k \equiv K/AL$, is constant on the balanced growth path, K must grow at the same rate as AL , which is $n + g$. Since the production function has constant returns to capital and effective labor, which both grow at rate $n + g$ on the balanced growth path, output must also grow at rate $n + g$ on the balanced growth path. Thus we have indeed found a balanced growth path where all the variables of the model grow at constant rates.

The next step is to show that the economy actually converges to this balanced growth path. At $k = k^*$, $f'(k) = (n + g + \delta)$. If $k > k^*$, $f'(k) < (n + g + \delta)$. This is because we are assuming that $f''(k) < 0$ meaning that $f'(k)$ falls as k rises. Thus if $k > k^*$, we have $\dot{k} < 0$ so that k will fall toward its balanced-

growth-path value. If $k < k^*$, $f'(k) > (n + g + \delta)$. This is because we are assuming that $f''(k) < 0$ meaning that $f'(k)$ rises as k falls. Thus if $k < k^*$, we have $\dot{k} > 0$ so that k will rise toward its balanced-growth-path value. Thus regardless of the initial value of k (as long as it is not $k = 0$), the economy will converge to a balanced growth path at k^* , where all the variables in the model are growing at a constant rate.

b) The golden-rule level of k — the level of k that maximizes consumption per unit of effective labor — is defined implicitly by $f'(k^{GR}) = (n + g + \delta)$, which can be interpreted graphically as the slope of the production function being equal to the slope of the break-even investment line. Note that this is exactly the level of k that the economy converges to here where all capital income is saved and all labor income is consumed.

In this model, we are saving capital's contribution to output, which is the marginal product of capital times the amount of capital. If that contribution exceeds break-even investment — $(n + g + \delta)k$ — k rises. If it is less than break-even investment, k falls. Thus k settles down to a point where saving — the marginal product of capital times k — equals break-even investment — $(n + g + \delta)k$. That is, the economy settles down to a point where $f'(k)k = (n + g + \delta)k$ or equivalently $f'(k) = (n + g + \delta)$.

Problem 1.9

a) Since the labor force grows exponentially at rate n — $\dot{L}(t) = nL(t)$ — we can write the labor force at time t as:

$$(1) L(t) = e^{nt} L(0)$$

Using equation (1), the amount of effective labor at time t is given by:

$$(2) c_L e^{gt} L(t) = c_L e^{gt} e^{nt} L(0) = c_L e^{(n+g)t} L(0)$$

Since $c_K K(0) = c_L L(0)$ and thus solving for $L(0)$:

$$(3) L(0) = [c_K K(0)]/c_L$$

Substituting equation (3) into equation (2) yields:

$$c_L e^{gt} L(t) = c_L e^{(n+g)t} [c_K K(0)]/c_L$$

or simply:

$$(4) c_L e^{gt} L(t) = c_K e^{(n+g)t} K(0)$$

Thus $c_K K(t) = c_L e^{(n+g)t} L(t)$ for all t is equivalent to $c_K K(t) = c_K e^{(n+g)t} K(0)$ for all t or simply:

$$(5) K(t) = e^{(n+g)t} K(0) \quad \text{for all } t$$

Equation (5) simply states that the capital stock grows exponentially at rate $(n + g)$. Thus

$c_K K(t) = c_L e^{(n+g)t} L(t)$ for all t is equivalent to:

$$(6) \dot{K}(t)/K(t) = n + g \quad \text{for all } t$$

Now, dividing the capital accumulation equation — $\dot{K}(t) = sY(t) - \delta K(t)$ — by $K(t)$ yields:

$$(7) \frac{\dot{K}(t)}{K(t)} = \frac{sY(t)}{K(t)} - \delta$$

Substituting the production function into equation (7) gives us:

$$(8) \frac{\dot{K}(t)}{K(t)} = \frac{s \cdot \min[c_K K(t), c_L e^{gt} L(t)]}{K(t)} - \delta$$

With $c_K K(t) = c_L e^{gt} L(t)$ for all t , we can write $\min[c_K K(t), c_L e^{gt} L(t)] = c_K K(t)$. Using this fact as well as equation (6), $c_K K(t) = c_L e^{gt} L(t)$ for all t is equivalent to:

$$(9) n + g = \frac{s c_K K(t)}{K(t)} - \delta$$

or simply:

$$(10) \quad n + g = s c_K - \delta$$

Thus $c_K K(t) = c_L e^{\delta t} L(t)$ for all t if the parameters are such that equation (10) holds. Clearly there is no particular reason to believe that equation (10) must hold.

b) If $c_L e^{\delta t} L(t)$ is growing faster than $c_K K(t)$, the amount of labor that exceeds $c_K K(t)$ will be unemployed. Since $c_K K(0) = c_L L(0)$, the difference between the quantity of effective labor and the quantity of capital is rising over time. That is, the quantity of unemployed people will be rising over time and the unemployment rate will approach 100%.

c) If $c_K K(t)$ is growing faster than $c_L e^{\delta t} L(t)$, the amount of capital that exceeds $c_L e^{\delta t} L(t)$ will not be utilized. Since $c_K K(0) = c_L L(0)$, the difference between the quantity of capital and the quantity of effective labor is rising over time. Thus the fraction of the capital stock that is not utilized will be rising over time.

Problem 1.10

a) We already have by assumption that $\dot{A}/A = g$, $\dot{L}/L = n$ and $\dot{R}/R = 0$. Thus we need to show that $\dot{K}/K$ and $\dot{Y}/Y$ converge to constants. Taking the time derivative of the log of the capital accumulation equation yields an expression for the growth rate of capital, g_K :

$$(1) \quad g_K \equiv \frac{\dot{K}}{K} = \frac{sY}{K} - \frac{\delta K}{K} = \frac{sY}{K} - \delta$$

Taking the time derivative of the log of the production function yields the following growth rate of output:

$$(2) \quad \frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + \beta \left(\frac{\dot{A}L}{AL} \right) + (1 - \alpha - \beta) \frac{\dot{R}}{R} = \alpha g_K + \beta(n + g)$$

As long as $\dot{K}/K$ is constant, then $\dot{Y}/Y$ will be constant as well. Thus we need to show that the economy converges to a situation where the growth rate of capital, $g_K \equiv \dot{K}/K$, is constant — that is, a situation where $\dot{g}_K \equiv \partial(\dot{K}/K)/\partial t$, the derivative of $(\dot{K}/K)$ with respect to time, is zero.

Taking the derivative of both sides of equation (1) with respect to time yields:

$$(3) \quad \dot{g}_K \equiv \frac{\partial(\dot{K}/K)}{\partial t} = s \cdot \left[\frac{\dot{Y}K - Y\dot{K}}{K^2} \right] = s \cdot \left[\frac{\dot{Y}}{K} - \frac{Y}{K} g_K \right]$$

Rearranging equation (2) to solve for the change in output gives us:

$$(4) \quad \dot{Y} = Y[\alpha g_K + \beta(n + g)]$$

Substituting equation (4) into equation (3) leaves us with:

$$(5) \quad \dot{g}_K = s \cdot \left[\frac{\alpha Y g_K + \beta Y(n + g)}{K} - \frac{Y}{K} g_K \right] = \frac{sY}{K} [(\alpha - 1)g_K + \beta(n + g)]$$

Rearranging equation (1) yields:

$$(6) \quad sY/K = g_K + \delta$$

Substituting equation (6) into equation (5) yields:

$$(7) \dot{g}_K = [g_K + \delta] \cdot [(\alpha - 1)g_K + \beta(n + g)]$$

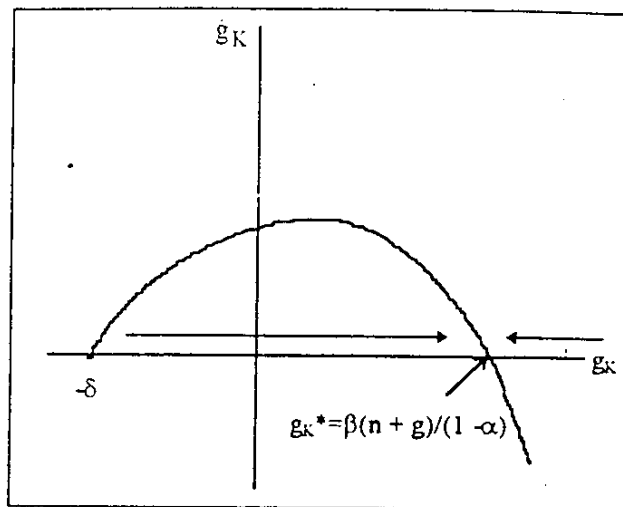
On a balanced growth path, we need:

$$(8) [g_K + \delta] \cdot [(\alpha - 1)g_K + \beta(n + g)] = 0$$

There are two possibilities:

$$1) g_K = -\delta \text{ and } 2) g_K = \beta(n + g)/(1 - \alpha)$$

The phase diagram implied by equation (7) is depicted at right. Equation (7) yields an inverted parabola with these two roots. We can see that $g_K^* = \beta(n + g)/(1 - \alpha)$ is a stable balanced growth path while $g_K^* = -\delta$ is unstable.



To obtain the growth rate of output on the balanced growth path, substitute the value of g_K^* into equation (2):

$$\frac{\dot{Y}}{Y} = \frac{\alpha\beta(n + g)}{(1 - \alpha)} + \beta(n + g) = \frac{\alpha\beta(n + g) + \beta(n + g) - \alpha\beta(n + g)}{1 - \alpha}$$

or simply:

$$(9) \frac{\dot{Y}}{Y} = \frac{\beta(n + g)}{1 - \alpha}$$

b) From equation (9), $\dot{Y}/Y$ will be positive so there will be permanent growth in output. It is also possible to come up with a condition on some of the parameters which ensures that output per person is also growing at a positive rate on the balanced growth path. For ease of notation, define $\gamma \equiv 1 - \alpha - \beta$ to be the elasticity of output with respect to land. Thus:

$$\frac{(\dot{Y}/L)}{(Y/L)} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \frac{\beta(n + g)}{1 - \alpha} - n = \frac{\beta(n + g)}{\beta + \gamma} - n$$

where the last step is rearranging $\alpha + \beta + \gamma = 1$ to replace $1 - \alpha$. Further simplification yields:

$$\frac{(\dot{Y}/L)}{(Y/L)} = \frac{\beta(n + g) - n(\beta + \gamma)}{\beta + \gamma} = \frac{\beta g - \gamma n}{\beta + \gamma}$$

So for positive growth in output per person we need:

$$\frac{\beta g - \gamma n}{\beta + \gamma} > 0 \quad \text{or} \quad \beta g - \gamma n > 0 \quad \text{or} \quad g > \frac{\gamma}{\beta} \cdot n$$

Thus the bigger is g (the higher is the growth rate of technology), the bigger is β (the more important effective labor is in production), the smaller is γ (the less important land is to the production process) and the smaller is n (the smaller the population growth rate), the more likely it is for there to be permanent growth in output per person. Basically, although the stock of land is fixed, as long as technology is improving fast enough, as long as land is not too important in the production process and as long as the number of people in the economy is not rising too quickly, the same amount of land can be used in a more productive way and the economy can experience long-run growth in output per person.

Problem 1.11

a) The production function with capital-augmenting technological progress is given by:

$$(1) Y(t) = [A(t)K(t)]^\alpha L(t)^{1-\alpha}$$

Dividing both sides of equation (1) by $A(t)^{\alpha/(1-\alpha)}L(t)$ yields:

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} \right]^\alpha \left[\frac{L(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} \right]^{1-\alpha}$$

and simplifying:

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}K(t)}{L(t)} \right]^\alpha \cdot A(t)^{-\alpha} = \left[\frac{A(t)^{1-\alpha/(1-\alpha)}A(t)^{-1}K(t)}{L(t)} \right]^\alpha$$

and thus finally:

$$\frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{K(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} \right]^\alpha$$

Now, defining $\phi \equiv \alpha/(1-\alpha)$, $k(t) \equiv K(t)/A(t)^\phi L(t)$ and $y(t) \equiv Y(t)/A(t)^\phi L(t)$ yields:

$$(2) y(t) = k(t)^\alpha$$

In order to analyze the dynamics of $k(t)$, take the time derivative of both sides of $k(t) \equiv K(t)/A(t)^\phi L(t)$:

$$\dot{k}(t) = \frac{\dot{K}(t)[A(t)^\phi L(t)] - K(t)[\phi A(t)^{\phi-1} \dot{A}(t)L(t) + \dot{L}(t)A(t)^\phi]}{[A(t)^\phi L(t)]^2}$$

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)^\phi L(t)} - \frac{K(t)}{A(t)^\phi L(t)} \left[\phi \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} \right]$$

and then using $k(t) \equiv K(t)/A(t)^\phi L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields:

$$(3) \dot{k}(t) = \dot{K}(t)/A(t)^\phi L(t) - (\phi\mu + n)k(t)$$

The evolution of the total capital stock is given by the usual:

$$(4) \dot{K}(t) = sY(t) - \delta K(t)$$

Substituting equation (4) into (3) gives us:

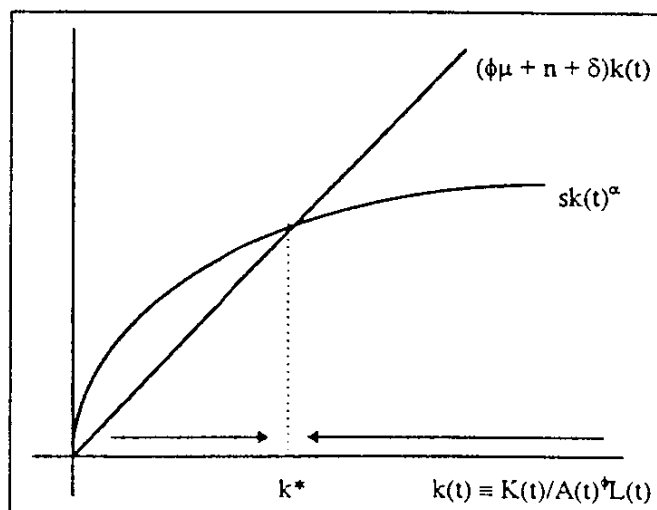
$$\dot{k}(t) = sY(t)/A(t)^\phi L(t) - \delta K(t)/A(t)^\phi L(t) - (\phi\mu + n)k(t) = sy(t) - (\phi\mu + n + \delta)k(t)$$

Finally, using equation (2) — $y(t) = k(t)^\alpha$ — we have:

$$(5) \dot{k}(t) = sk(t)^\alpha - (\phi\mu + n + \delta)k(t)$$

Equation (5) is very similar to the basic equation governing the dynamics of the Solow model with labor-augmenting technological progress. Here, however, everything is in units of $A(t)^\phi L(t)$ rather than in units of effective labor, $A(t)L(t)$. Using the same graphical technique as with the basic Solow model, we can graph both components of $\dot{k}(t)$. See the diagram at right.

When actual investment per unit of $A(t)^\phi L(t)$ — $sk(t)^\alpha$ — exceeds break-even investment per unit of $A(t)^\phi L(t)$ — $(\phi\mu + n + \delta)k(t)$ — k will rise towards k^* . When actual investment per unit of $A(t)^\phi L(t)$ falls short of break-even investment



per unit of $A(t)^\phi L(t)$, k will fall towards k^* . The economy will converge to a situation where k is constant at k^* (ignoring the possibility that the initial level is k is zero). Since $y = k^\alpha$, y will also be constant when the economy converges to k^* .

The total capital stock, K , can be written as $A^\phi Lk$. Thus when k is constant, K will be growing at the constant rate of $\phi\mu + n$. Similarly, total output, Y , can be written as $A^\phi Ly$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$ as well. Since L and A grow at constant rates by assumption, we have found a balanced growth path where all the variables of the model grow at constant rates.

b) The production function is now given by:

$$(6) Y(t) = J(t)^\alpha L(t)^{1-\alpha}$$

Define $\bar{J}(t) \equiv J(t)/A(t)$. The production function can then be written as:

$$(7) Y(t) = [A(t)\bar{J}(t)]^\alpha L(t)^{1-\alpha}$$

Proceed as in part (a). Divide both sides of equation (7) by $A(t)^{\alpha(1-\alpha)}L(t)$ and simplify to obtain:

$$(8) \frac{Y(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} = \left[\frac{\bar{J}(t)}{A(t)^{\alpha/(1-\alpha)}L(t)} \right]^\alpha$$

Now, defining $\phi \equiv \alpha/(1-\alpha)$, $\bar{j}(t) \equiv \bar{J}(t)/A(t)^\phi L(t)$ and $y(t) \equiv Y(t)/A(t)^\phi L(t)$ yields:

$$(9) y(t) = \bar{j}(t)^\alpha$$

In order to analyze the dynamics of $\bar{j}(t)$, take the time derivative of both sides of $\bar{j}(t) \equiv \bar{J}(t)/A(t)^\phi L(t)$:

$$\dot{\bar{j}} = \frac{\dot{\bar{J}}[A(t)^\phi L(t)] - \bar{J}[\phi A(t)^{\phi-1} \dot{A}(t)L(t) + \dot{L}(t)A(t)^\phi]}{[A(t)^\phi L(t)]^2}$$

$$\dot{\bar{j}} = \frac{\dot{\bar{J}}}{A(t)^\phi L(t)} - \frac{\bar{J}}{A(t)^\phi L(t)} \left[\phi \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} \right]$$

and then using $\bar{j}(t) \equiv \bar{J}(t)/A(t)^\phi L(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{L}(t)/L(t) = n$ yields:

$$(10) \dot{\bar{j}} = \bar{j} / A(t)^\phi L(t) - (\phi\mu + n)\bar{j}(t)$$

The next step is to get an expression for $\dot{\bar{J}}(t)$. Take the time derivative of both sides of $\bar{J}(t) \equiv J(t)/A(t)$:

$$\dot{\bar{J}}(t) = \frac{\dot{J}(t)A(t) - J(t)\dot{A}(t)}{A(t)^2} = \frac{\dot{J}(t)}{A(t)} - \frac{\dot{A}(t)}{A(t)} \cdot \frac{J(t)}{A(t)}$$

Now use $\bar{J}(t) \equiv J(t)/A(t)$, $\dot{A}(t)/A(t) = \mu$ and $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$ to obtain:

$$\dot{\bar{J}}(t) = \frac{sA(t)Y(t)}{A(t)} - \frac{\delta J(t)}{A(t)} - \mu \bar{J}(t)$$

or simply:

$$(11) \dot{\bar{J}}(t) = sY(t) - (\mu + \delta)\bar{J}(t)$$

Substitute equation (11) into equation (10):

$$\dot{\bar{j}}(t) = sY(t)/A(t)^\phi L(t) - (\mu + \delta)\bar{j}(t)/A(t)^\phi L(t) - (\phi\mu + n)\bar{j}(t) = sy(t) - [n + \delta + \mu(1 + \phi)]\bar{j}(t)$$

Finally, using equation (9) — $y(t) = \bar{j}(t)^\alpha$ — we have:

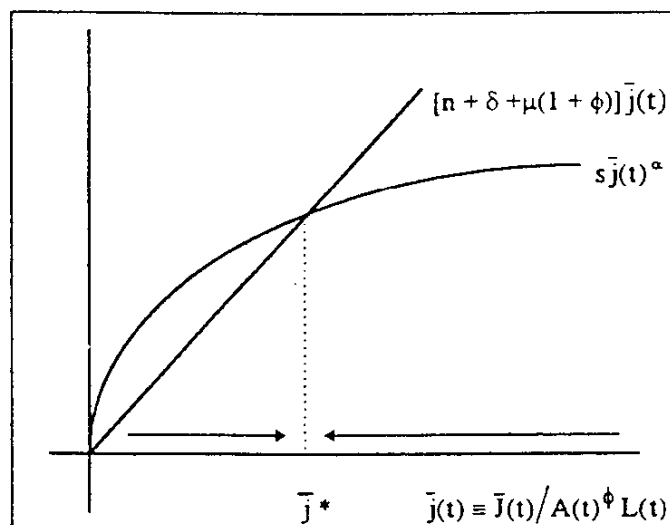
$$(12) \dot{\bar{j}}(t) = s\bar{j}(t)^\alpha - [n + \delta + \mu(1 + \phi)]\bar{j}(t)$$

Using the same graphical technique as with the basic Solow model, we can graph both components of $\dot{\bar{j}}(t)$.

See the diagram at right. The economy will converge to a situation where $\bar{j}$ is constant at $\bar{j}^*$ (ignoring the possibility that the initial level is $\bar{j} = 0$). Since $y = \bar{j}^\alpha$, y will also be constant when the economy converges to $\bar{j}^*$.

The level of total output, Y , can be written as $A^\phi L y$. Thus when y is constant, output grows at the constant rate of $\phi\mu + n$.

By definition, $\bar{J} \equiv A^\phi L \bar{j}$. Once the economy converges to the situation where $\bar{j}$ is constant, $\bar{J}$ grows at the constant rate of $\phi\mu + n$. Since $J \equiv \bar{J} \cdot A$, the effective capital stock, J , grows at rate $\phi\mu + n + \mu$ or simply $n + \mu(1 + \phi)$. Thus the economy does converge to a balanced growth path where all the variables of the model are growing at constant rates.



c) On the balanced growth path, $\dot{j}(t) = 0$ and thus from equation (12):

$$s \bar{j}^\alpha = [n + \delta + \mu(1 + \phi)] \bar{j} \Rightarrow \bar{j}^{1-\alpha} = s / [n + \delta + \mu(1 + \phi)]$$

and thus:

$$(13) \quad \bar{j}^* = [s / (n + \delta + \mu(1 + \phi))]^{1/(1-\alpha)}$$

Substitute equation (13) into equation (9) to get an expression for output per unit of $A(t)^\phi L(t)$ on the balanced growth path:

$$(14) \quad y^* = [s / (n + \delta + \mu(1 + \phi))]^{\alpha/(1-\alpha)}$$

Take the derivative of y^* with respect to s :

$$\frac{\partial y^*}{\partial s} = \left[\frac{\alpha}{1-\alpha} \right] \cdot \left[\frac{s}{n + \delta + \mu(1 + \phi)} \right]^{\alpha/(1-\alpha)-1} \cdot \left[\frac{1}{n + \delta + \mu(1 + \phi)} \right]$$

In order to turn this into an elasticity, multiply both sides by s/y^* using the expression for y^* from equation (14) on the right-hand side:

$$\frac{\partial y^*}{\partial s} \cdot \frac{s}{y^*} = \left[\frac{\alpha}{1-\alpha} \right] \cdot \left[\frac{s}{n + \delta + \mu(1 + \phi)} \right]^{\alpha/(1-\alpha)-1} \cdot \left[\frac{1}{n + \delta + \mu(1 + \phi)} \right] \cdot s \cdot \left[\frac{s}{n + \delta + \mu(1 + \phi)} \right]^{-\alpha/(1-\alpha)}$$

simplifying yields:

$$\frac{\partial y^*}{\partial s} \cdot \frac{s}{y^*} = \left[\frac{\alpha}{1-\alpha} \right] \cdot \left[\frac{n + \delta + \mu(1 + \phi)}{s} \right] \cdot \left[\frac{s}{n + \delta + \mu(1 + \phi)} \right]$$

and thus finally:

$$(15) \quad \frac{\partial y^*}{\partial s} \cdot \frac{s}{y^*} = \frac{\alpha}{1-\alpha}$$

d) A first-order Taylor approximation of $\dot{y}$ around the balanced-growth-path value of $y = y^*$ will be of the form:

$$(16) \quad \dot{y} \cong \left. \frac{\partial \dot{y}}{\partial y} \right|_{y=y^*} \cdot [y - y^*]$$

Taking the time derivative of both sides of equation (9) yields:

$$(17) \quad \dot{y} = \alpha \bar{j}^{\alpha-1} \dot{\bar{j}}$$

Substitute equation (12) into equation (17):

$$\dot{y} = \alpha \bar{j}^{\alpha-1} \cdot [s \bar{j}^\alpha - (n + \delta + \mu(1 + \phi)) \bar{j}]$$

or:

$$(18) \quad \dot{y} = s \alpha \bar{j}^{2\alpha-1} - \alpha \bar{j}^\alpha \cdot [n + \delta + \mu(1 + \phi)]$$

Equation (18) expresses $\dot{y}$ in terms of $\bar{j}$. We can express $\bar{j}$ in terms of y : since $y = \bar{j}^\alpha$, we can write $\bar{j} = y^{1/\alpha}$. Thus $\partial \dot{y} / \partial y$ evaluated at $y = y^*$ is given by:

$$\left. \frac{\partial \dot{y}}{\partial y} \right|_{y=y^*} = \left[\left. \frac{\partial \dot{y}}{\partial \bar{j}} \right|_{y=y^*} \right] \cdot \left[\left. \frac{\partial \bar{j}}{\partial y} \right|_{y=y^*} \right] = [s \alpha (2\alpha - 1) \bar{j}^{2(\alpha-1)} - \alpha \bar{j}^{2\alpha-1} (n + \delta + \mu(1 + \phi))] \cdot \left[\frac{1}{\alpha} \cdot y^{(1-\alpha)/\alpha} \right]$$

Now, $y^{(1-\alpha)/\alpha}$ is simply $\bar{j}^{1-\alpha}$ since $y = \bar{j}^\alpha$ and thus:

$$\left. \frac{\partial \dot{y}}{\partial y} \right|_{y=y^*} = s(2\alpha - 1) \bar{j}^{2(\alpha-1)+(1-\alpha)} - \alpha \bar{j}^{\alpha-1+(1-\alpha)} [n + \delta + \mu(1 + \phi)] = s(2\alpha - 1) \bar{j}^{\alpha-1} - \alpha [n + \delta + \mu(1 + \phi)]$$

Finally, substitute out for s by rearranging equation (13) to obtain $s = \bar{j}^{1-\alpha} [n + \delta + \mu(1 + \phi)]$ and thus:

$$\left. \frac{\partial \dot{y}}{\partial y} \right|_{y=y^*} = \bar{j}^{1-\alpha} [n + \delta + \mu(1 + \phi)] (2\alpha - 1) \bar{j}^{\alpha-1} - \alpha [n + \delta + \mu(1 + \phi)]$$

or simply:

$$(19) \quad \left. \frac{\partial \dot{y}}{\partial y} \right|_{y=y^*} = -(1 - \alpha) [n + \delta + \mu(1 + \phi)]$$

Substituting equation (19) into equation (16) gives the first-order Taylor expansion:

$$(20) \quad \dot{y} \cong -(1 - \alpha) [n + \delta + \mu(1 + \phi)] \cdot [y - y^*]$$

Solving this differential equation (as in the text) yields:

$$(21) \quad y(t) - y^* = e^{-(1-\alpha)[n+\delta+\mu(1+\phi)]t} \cdot [y(0) - y^*]$$

This means that the economy moves fraction $(1 - \alpha) \cdot [n + \delta + \mu(1 + \phi)]$ of the remaining distance towards y^* each year.

e) The elasticity of output with respect to s is the same in this model as in the basic Solow model. The speed of convergence is faster in this model. In the basic Solow model, the rate of convergence is given by $(1 - \alpha) \cdot [n + \delta + \mu]$ whereas in this model it is $(1 - \alpha) \cdot [n + \delta + \mu(1 + \phi)]$ which is greater since $\phi \equiv \alpha / (1 - \alpha)$ is positive.

Problem 1.12

a) The growth-accounting techniques of Section 1.7 yielded the following expression for the growth rate of output per person:

$$(1) \quad \frac{\dot{Y}(t)}{Y(t)} - \frac{\dot{L}(t)}{L(t)} = \alpha_K(t) \left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} \right] + R(t)$$

where $\alpha_K(t)$ is the elasticity of output with respect to capital at time t and $R(t)$ is the Solow residual. Now imagine applying this growth-accounting equation to a Solow economy that is on its balanced growth path. On the balanced growth path, the growth rates of output per worker and capital per worker are both equal to g , the growth rate of A . Thus equation (1) implies that growth accounting would attribute a fraction α_K of growth in output per worker to growth in capital per worker. It would attribute the rest — fraction $1 - \alpha_K$ — to technological progress, as this is what would be left in the Solow residual. So with our usual estimate of $\alpha_K = 1/3$, growth accounting would attribute about 67% of the growth in output per worker to technological progress and about 33% of the growth in output per worker to growth in capital per worker.

b) In an accounting sense, the result in part (a) would be true, but in a deeper sense it would not: the reason that the capital-labor ratio grows at rate g on the balanced growth path is because the effectiveness of labor is growing at rate g . That is, the growth in the effectiveness of labor — the growth in A — raises output per worker through two channels. First, by directly raising output but also by (for a given saving rate) increasing the resources devoted to capital accumulation and thereby raising the capital-labor ratio. Growth accounting attributes the rise in output per worker through the second channel to growth in the capital-labor ratio, and not to its underlying source. Thus, although growth accounting is often instructive, it is not appropriate to interpret it as shedding light on the underlying determinants of growth.

Problem 1.13

a) OLS will yield a biased estimate of the slope coefficient of a regression if the explanatory variable is correlated with the error term. We are given that :

$$(1) \ln \left[\left(\frac{Y}{N} \right)_{1979} \right] - \ln \left[\left(\frac{Y}{N} \right)_{1870} \right]^* = a + b \cdot \ln \left[\left(\frac{Y}{N} \right)_{1870} \right]^* + \varepsilon$$

$$(2) \ln \left[\left(\frac{Y}{N} \right)_{1870} \right] = \ln \left[\left(\frac{Y}{N} \right)_{1870} \right]^* + u$$

where ε and u are assumed to be uncorrelated with each other and with the true unobservable 1870 income per person variable, $\ln[(Y/N)_{1870}]^*$.

Substituting equation (2) into (1) and rearranging yields :

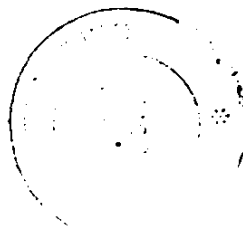
$$(3) \ln \left[\left(\frac{Y}{N} \right)_{1979} \right] - \ln \left[\left(\frac{Y}{N} \right)_{1870} \right] = a + b \cdot \ln \left[\left(\frac{Y}{N} \right)_{1870} \right] + [\varepsilon - (1+b)u]$$

Running an OLS regression on model (3) will yield a biased estimate of b if $\ln[(Y/N)_{1870}]$ is correlated with the error term, $[\varepsilon - (1+b)u]$. In general, of course, this will be the case since u is the measurement error that helps to determine the value of $\ln[(Y/N)_{1870}]$ that we get to observe. However, in the special case where the true value of $b = -1$, the error term in model (3) is simply ε . Thus OLS will be unbiased since the explanatory variable will no longer be correlated with the error term.

b) Measurement error in the dependent variable will not cause a problem for OLS estimation and is, in fact, one of the justifications for the disturbance term in a regression model. Intuitively, if the measurement error is in 1870 income (per capita), the explanatory variable, there will be a bias towards finding convergence. If 1870 income is overstated, growth is understated. This looks like convergence — a "high" initial income country growing slowly. Similarly, if 1870 income is understated, growth is overstated. This also looks like convergence — a "low" initial income country growing quickly.

Suppose instead that it is only 1979 income (per capita) that is subject to random, mean -zero measurement error. When 1979 income is overstated, so is growth for a given level of 1870 income. When 1979 income is understated, so is growth for a given 1870 income. Either case is equally likely — overstating 1979 income for any given 1870 income is just as likely as understating it (or more precisely, measurement error is on average equal to zero). Thus there is no reason for this to systematically cause us to see more or less convergence than there really is in the data.

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SOLUTIONS TO CHAPTER 2

Problem 2.1

a) The firm's problem is to choose the quantities of capital, K , and effective labor, AL , in order to minimize costs -- $wAL + rK$ -- subject to the production function, $Y = ALf(k)$. Set up the Lagrangian:

$$\mathcal{L} = wAL + rK + \lambda[Y - ALf(K/AL)]$$

The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda[AL \cdot f'(K/AL) \cdot (1/AL)] = 0 \quad \Rightarrow \quad r = \lambda f'(k) \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial (AL)} = w - \lambda[f(K/AL) + AL \cdot f'(K/AL) \cdot (-K)/(AL)^2] = 0 \quad \Rightarrow \quad w = \lambda[f(k) - k \cdot f'(k)] \quad (2)$$

Dividing equation (1) by equation (2) gives us:

$$(3) \quad \frac{r}{w} = \frac{f'(k)}{f(k) - k \cdot f'(k)}$$

Equation (3) implicitly defines the cost minimizing choice of k . Clearly this choice does not depend upon the level of output, Y . Note that equation (3) is the standard cost-minimizing condition which is that the ratio of the marginal cost of the two inputs, capital and effective labor, must equal the ratio of the marginal products of the two inputs.

b) Since, as shown in part (a), each firm chooses the same value of k and since we are told that each firm has the same value of A , we can write the total amount produced by the N cost-minimizing firms as:

$$\sum_{i=1}^n Y_i = \sum_{i=1}^n AL_i f(k) = Af(k) \sum_{i=1}^n L_i = A\bar{L}f(k)$$

where $\bar{L}$ is the total amount of labor employed.

The single firm also has the same value of A and would choose the same value of k -- the choice of k does not depend on Y . Thus if it used all of the labor employed by the N cost-minimizing firms, $\bar{L}$, the single firm would produce $Y = A\bar{L}f(k)$. This is exactly the same amount of output produced in total by the N cost-minimizing firms.

Problem 2.2

a) The individual's problem is to maximize lifetime utility given by:

$$(1) \quad U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \cdot \frac{C_2^{1-\theta}}{1-\theta}$$

subject to the lifetime budget constraint given by:

$$(2) \quad P_1 C_1 + P_2 C_2 = W$$

where W represents lifetime income.

Rearrange the budget constraint to solve for C_2 :

$$(3) \quad C_2 = W/P_2 - C_1 \cdot P_1/P_2$$

Substitute equation (3) into equation (1):

$$(4) \quad U = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \cdot \frac{[W/P_2 - C_1 \cdot P_1/P_2]^{1-\theta}}{1-\theta}$$

Now we can solve the unconstrained problem of maximizing utility -- as given by equation (4) -- with respect to first-period consumption, C_1 . The first-order condition is given by:

$$\partial U / \partial C_1 = C_1^{-\theta} + (1/\rho) \cdot C_2^{-\theta} \cdot (-P_1/P_2) = 0 \Rightarrow C_1^{-\theta} = (1/\rho) \cdot (P_1/P_2) \cdot C_2^{-\theta}$$

or simply:

$$(5) C_1 = (1+\rho)^{1/\theta} \cdot (P_2/P_1)^{1/\theta} \cdot C_2$$

In order to solve for C_2 , substitute equation (5) into equation (3):

$$C_2 = W/P_2 - (1+\rho)^{1/\theta} \cdot (P_2/P_1)^{1/\theta} \cdot C_2 \cdot (P_1/P_2) \Rightarrow C_2 \left[1 + (1+\rho)^{1/\theta} \cdot (P_2/P_1)^{(1-\theta)/\theta} \right] = W/P_2$$

or simply:

$$(6) C_2 = \frac{W/P_2}{1 + (1+\rho)^{1/\theta} \cdot (P_2/P_1)^{(1-\theta)/\theta}}$$

Finally, to get the optimal choice of C_1 , substitute equation (6) into equation (5):

$$(7) C_1 = \frac{(1+\rho)^{1/\theta} \cdot (P_2/P_1)^{1/\theta} \cdot (W/P_2)}{1 + (1+\rho)^{1/\theta} \cdot (P_2/P_1)^{(1-\theta)/\theta}}$$

b) From equation (5), the optimal ratio of first-period to second-period consumption is:

$$(8) C_1/C_2 = (1+\rho)^{1/\theta} \cdot (P_2/P_1)^{1/\theta}$$

Taking the log of both sides of equation (8) yields:

$$(9) \ln(C_1/C_2) = (1/\theta) \cdot \ln(1+\rho) + (1/\theta) \cdot \ln(P_2/P_1)$$

The elasticity of substitution between C_1 and C_2 – defined in such a way that it is positive – is given by:

$$\frac{\partial(C_1/C_2)}{\partial(P_2/P_1)} \cdot \frac{(P_2/P_1)}{(C_1/C_2)} = \frac{\partial[\ln(C_1/C_2)]}{\partial[\ln(P_2/P_1)]} = \frac{1}{\theta}$$

where the last step uses equation (9) to find the derivative. Thus higher values of θ imply that the individual is less willing to substitute consumption between periods.

Problem 2.3

We need to solve the household's problem keeping everything in per capita terms rather than in units of effective labor and assuming log utility. The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize:

$$(1) U = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt$$

subject to:

$$(2) \int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt = \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt$$

$$\text{Now let } W \equiv \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)} A(t) w(t) \frac{L(t)}{H} dt$$

We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t} \ln C(t) \frac{L(t)}{H} dt + \lambda \cdot \left[W - \int_{t=0}^{\infty} e^{-R(t)} C(t) \frac{L(t)}{H} dt \right]$$

The first-order condition is given by:

$$\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t} C(t)^{-1} \frac{L(t)}{H} - \lambda e^{-R(t)} \frac{L(t)}{H} = 0$$

Canceling the $L(t)/H$ yields:

$$(3) e^{-\rho t} C(t)^{-1} = \lambda e^{-R(t)}$$

which implies:

$$(4) C(t) = e^{-\rho t} \lambda^{-1} e^{R(t)}$$

Substituting this into the budget constraint, equation (2), gives us:

$$(5) \int_{t=0}^{\infty} e^{-R(t)} \left[e^{-\rho t} \lambda^{-1} e^{R(t)} \right] \frac{L(t)}{H} dt = W$$

Since $L(t) = e^{nt} L(0)$, this implies:

$$(6) \lambda^{-1} \frac{L(0)}{H} \int_{t=0}^{\infty} e^{-(\rho-n)t} dt = W$$

As long as $\rho - n > 0$ (which it must be), the integral is equal to $1/(\rho - n)$ and thus λ^{-1} is given by:

$$(7) \lambda^{-1} = \frac{W}{L(0)/H} \cdot (\rho - n)$$

Substituting equation (7) into equation (4) yields:

$$(8) C(t) = e^{R(t)-\rho t} \cdot \left[\frac{W}{L(0)/H} \cdot (\rho - n) \right]$$

Initial consumption is therefore:

$$(9) C(0) = \frac{W}{L(0)/H} \cdot (\rho - n)$$

Note that $C(0)$ is consumption per person, W is wealth per household and $L(0)/H$ is the number of people per household. Thus $W/[L(0)/H]$ is wealth per person. This equation says that initial consumption per person is a constant fraction of initial wealth per person, and $(\rho - n)$ can be interpreted as the marginal propensity to consume out of wealth. With logarithmic utility, this propensity to consume is independent of the path of the real interest rate. Also note that the bigger is the household's discount rate ρ — the more the household discounts the future — the bigger is the fraction of wealth that it initially consumes.

Problem 2.4

The household's problem is to maximize lifetime utility subject to the budget constraint. That is, its problem is to maximize:

$$(1) U = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \cdot \frac{L(t)}{H} dt$$

subject to:

$$(2) \int_{t=0}^{\infty} e^{-rt} C(t) \frac{L(t)}{H} dt = W$$

where W denotes the household's initial wealth plus the present value of its lifetime labor income — the right-hand side of equation (2.5) in the text. Note that the real interest rate, r , is assumed to be constant. We can use the informal method, presented in the text, for solving this type of problem. Set up the Lagrangian:

$$\mathcal{L} = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \cdot \frac{L(t)}{H} dt + \lambda \cdot \left[W - \int_{t=0}^{\infty} e^{-rt} C(t) \frac{L(t)}{H} dt \right]$$

The first-order condition is given by:

$$\frac{\partial \mathcal{L}}{\partial C(t)} = e^{-\rho t} C(t)^{-\theta} \frac{L(t)}{H} - \lambda e^{-\pi} \frac{L(t)}{H} = 0$$

Canceling the $L(t)/H$ yields:

$$(3) \quad e^{-\rho t} C(t)^{-\theta} = \lambda e^{-\pi}$$

Differentiate both sides of equation (3) with respect to time:

$$e^{-\rho t} [-\theta C(t)^{-\theta-1} \dot{C}(t)] - \rho e^{-\rho t} C(t)^{-\theta} + r \lambda e^{-\pi} = 0$$

This can be rearranged to obtain:

$$(4) \quad -\theta \frac{\dot{C}(t)}{C(t)} e^{-\rho t} C(t)^{-\theta} - \rho e^{-\rho t} C(t)^{-\theta} + r \lambda e^{-\pi} = 0$$

Now substitute the first-order condition -- equation (3) -- into equation (4):

$$-\theta \frac{\dot{C}(t)}{C(t)} \lambda e^{-\pi} - \rho \lambda e^{-\pi} + r \lambda e^{-\pi} = 0$$

Canceling the $\lambda e^{-\pi}$ and solving for the growth rate of consumption, $\dot{C}(t)/C(t)$, yields:

$$(5) \quad \frac{\dot{C}(t)}{C(t)} = \frac{r - \rho}{\theta}$$

Thus with a constant real interest rate, the growth rate of consumption is a constant. If $r > \rho$ -- that is, if the rate that the market pays to defer consumption exceeds the household's discount rate -- consumption will be rising over time. The value of θ determines how fast consumption will grow if r exceeds ρ . Lower values of θ -- and thus higher values of the elasticity of substitution, $1/\theta$ -- means that consumption growth will be higher for any given difference between r and ρ .

We now need to solve for the path of $C(t)$. First, note that equation (5) can be rewritten as:

$$(6) \quad \frac{\partial \ln C(t)}{\partial t} = \frac{r - \rho}{\theta}$$

Integrate equation (6) forward from time $\tau = 0$ to time $\tau = t$:

$$\ln C(t) - \ln C(0) = \left[(r - \rho)/\theta \right] \tau \Big|_{\tau=0}^{\tau=t}$$

which simplifies to:

$$(7) \quad \ln[C(t)/C(0)] = [(r - \rho)/\theta] t$$

Taking the exponential function of both sides of equation (7) yields:

$$C(t)/C(0) = e^{[(r - \rho)/\theta] t}$$

and thus:

$$(8) \quad C(t) = C(0) \cdot e^{[(r - \rho)/\theta] t}$$

We can now solve for initial consumption, $C(0)$, by using the fact that it must be chosen to satisfy the household's budget constraint. Substitute equation (8) into equation (2):

$$\int_{t=0}^{\infty} e^{-\pi} C(0) e^{[(r - \rho)/\theta] t} \cdot \frac{L(t)}{H} dt = W$$

Using the fact that $L(t) = L(0)e^{nt}$ yields:

$$(9) \quad \frac{C(0)L(0)}{H} \int_{t=0}^{\infty} e^{-[\rho - r + \theta(r - \rho)]t/\theta} dt = W$$

As long as $[\rho - r + \theta(r - n)]/\theta > 0$, we can solve the integral.

$$(10) \int_{t=0}^{\infty} e^{-[\rho - r + \theta(r - n)]t/\theta} dt = \frac{\theta}{\rho - r + \theta(r - n)}$$

Substitute equation (10) into equation (9) and solve for $C(0)$:

$$(11) C(0) = \frac{W}{L(0)/H} \cdot \left[\frac{(\rho - r)}{\theta} + (r - n) \right]$$

Finally, to get an expression for consumption at each instant in time, substitute equation (11) into equation (8):

$$(12) C(t) = e^{[(r - \rho)/\theta]t} \cdot \frac{W}{L(0)/H} \cdot \left[\frac{(\rho - r)}{\theta} + (r - n) \right]$$

Problem 2.5

a) The equation describing the dynamics of the capital stock per unit of effective labor is:

$$(1) \dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$$

For a given k , the level of c that implies $\dot{k} = 0$ is given by $c = f(k) - (n + g)k$. Thus a fall in g makes the level of c consistent with $\dot{k} = 0$ higher for a given k . That is, the $\dot{k} = 0$ curve shifts up. Intuitively, a lower g makes break-even investment lower at any given k and thus allows for more resources to be devoted to consumption and still maintain a given k . Since $(n + g)k$ falls proportionately more at higher levels of k , the $\dot{k} = 0$ curve shifts up more at higher levels of k . See the diagram.

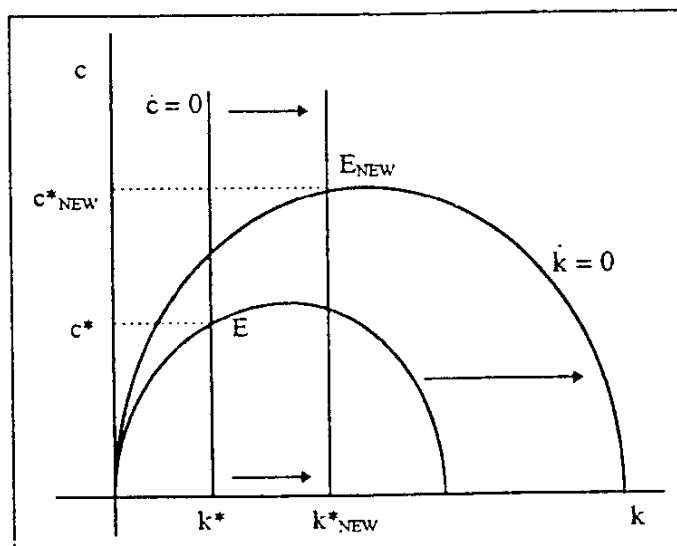
b) The equation describing the dynamics of consumption per unit of effective labor is given by:

$$(2) \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

Thus the condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. After the fall in g , $f'(k)$ must be lower in order for $\dot{c} = 0$. Since $f''(k)$ is negative this means that the k needed for $\dot{c} = 0$ therefore rises. Thus the $\dot{c} = 0$ curve shifts to the right.

c) At the time of the change in g , the value of k — the stock of capital per unit of effective labor — is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k^* on the old balanced growth path.

In contrast, c — the rate at which households are consuming in units of effective labor — can jump at the time of the shock. In order for the economy to reach the new balanced growth path, c must jump at the instant of the change so that the economy is on the new saddle path.



However, we cannot tell whether the new saddle path passes above or below the original point E . Thus we cannot tell whether c jumps up or down and in fact — if the new saddle path passes right through point E —

c might even remain the same at the instant that g falls. Thereafter, c and k rise gradually to their new balanced-growth-path values; these are higher than their values on the original balanced growth path.

d) On a balanced growth path, the fraction of output that is saved and invested is given by $[f(k^*) - c^*]/f(k^*)$. Since k is constant, or $\dot{k} = 0$ on a balanced growth path then -- from equation (1) -- we can write $f(k^*) - c^* = (n + g)k^*$. Using this, we can rewrite the fraction of output that is saved on a balanced growth path as:

$$(3) s = [(n + g)k^*]/f(k^*)$$

Differentiating both sides of equation (3) with respect to g yields:

$$(4) \frac{\partial s}{\partial g} = \frac{f(k^*)[(n + g)(\partial k^*/\partial g) + k^*] - (n + g)k^* f'(k^*)(\partial k^*/\partial g)}{[f(k^*)]^2}$$

which simplifies to:

$$(5) \frac{\partial s}{\partial g} = \frac{(n + g)[f(k^*) - k^* f'(k^*)](\partial k^*/\partial g) + f(k^*)k^*}{[f(k^*)]^2}$$

Since k^* is defined by $f'(k^*) = \rho + \theta g$, differentiating both sides of this expression gives us $f''(k^*)(\partial k^*/\partial g) = \theta$. Solving for $\partial k^*/\partial g$ gives us:

$$(6) \partial k^*/\partial g = \theta/f''(k^*) < 0$$

Substituting equation (6) into equation (5) yields:

$$(7) \frac{\partial s}{\partial g} = \frac{(n + g)[f(k^*) - k^* f'(k^*)]\theta + f(k^*)k^* f'(k^*)}{[f(k^*)]^2 f''(k^*)}$$

The first term in the numerator is positive, while the second is negative and so the sign of $\partial s/\partial g$ is ambiguous. Thus we cannot tell whether the fall in g raises or lowers the saving rate on the new balanced growth path.

e) When the production function is Cobb-Douglas, $f(k) = k^\alpha$, $f'(k) = \alpha k^{\alpha-1}$ and $f''(k) = \alpha(\alpha - 1)k^{\alpha-2}$.

Substituting these facts into equation (7) yields:

$$(8) \frac{\partial s}{\partial g} = \frac{(n + g)[k^{\alpha} - k^{\alpha} \alpha k^{\alpha-1}] \theta + k^{\alpha} k^{\alpha} \alpha(\alpha - 1) k^{\alpha-2}}{k^{\alpha} k^{\alpha} \alpha(\alpha - 1) k^{\alpha-2}}$$

which simplifies to:

$$(9) \frac{\partial s}{\partial g} = \frac{(n + g)k^{\alpha} (1 - \alpha)\theta - (1 - \alpha)k^{\alpha} \alpha k^{\alpha-1}}{[-(1 - \alpha)k^{\alpha} (\alpha k^{\alpha-1}) (\alpha k^{\alpha-1}) / \alpha]}$$

which implies:

$$(10) \frac{\partial s}{\partial g} = \alpha \frac{[(n + g)\theta - (\rho + \theta g)]}{(\rho + \theta g)^2}$$

Thus, finally, we have:

$$(11) \frac{\partial s}{\partial g} = \alpha \frac{(n - \rho)}{(\rho + \theta g)^2}$$

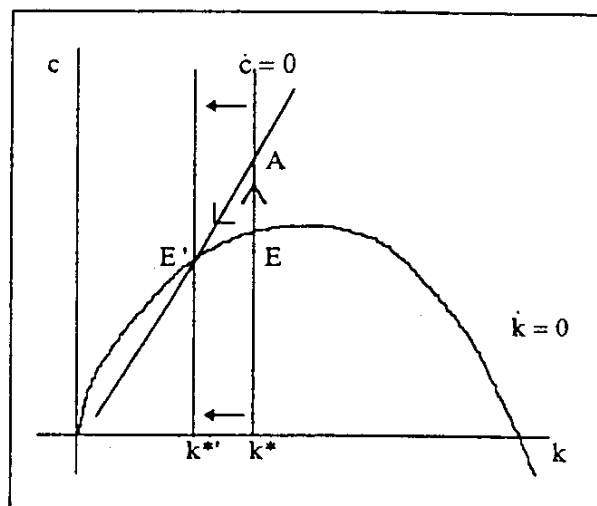
Problem 2.6

The two equations of motion are:

$$(1) \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta} \quad \text{and} \quad (2) \dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$$

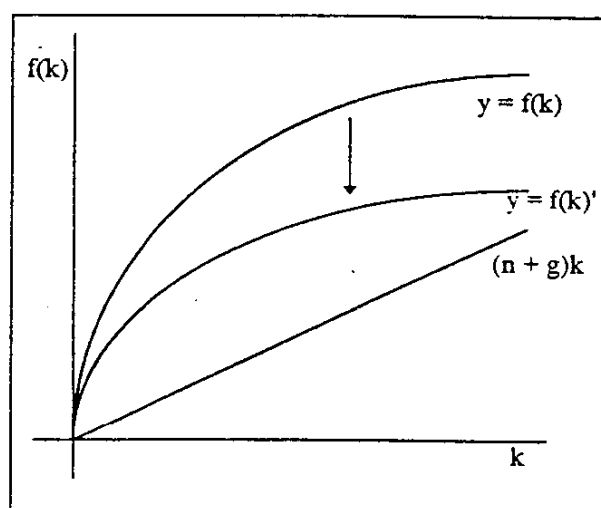
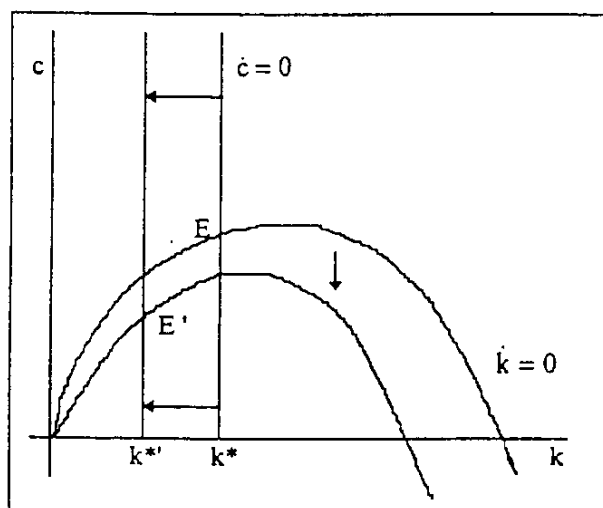
a) A rise in θ . This is a fall in the elasticity of substitution $-1/\theta$ —which means that people become less willing to substitute consumption between periods. It also means that the marginal utility of consumption falls off more rapidly as consumption rises. If the economy is growing, this tends to make households value present consumption more than future consumption.

The capital-accumulation equation is unaffected. The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. Since $f''(k) < 0$, the $f'(k)$ that makes $\dot{c} = 0$ is now higher. Thus the value of k that satisfies $\dot{c} = 0$ is lower. The $\dot{c} = 0$ locus shifts to the left. The economy moves up to point A on the new saddle path—people consume more now. Movement is then down along the new saddle path until the economy reaches point E'. At that point, c^* and k^* are lower than their original values.



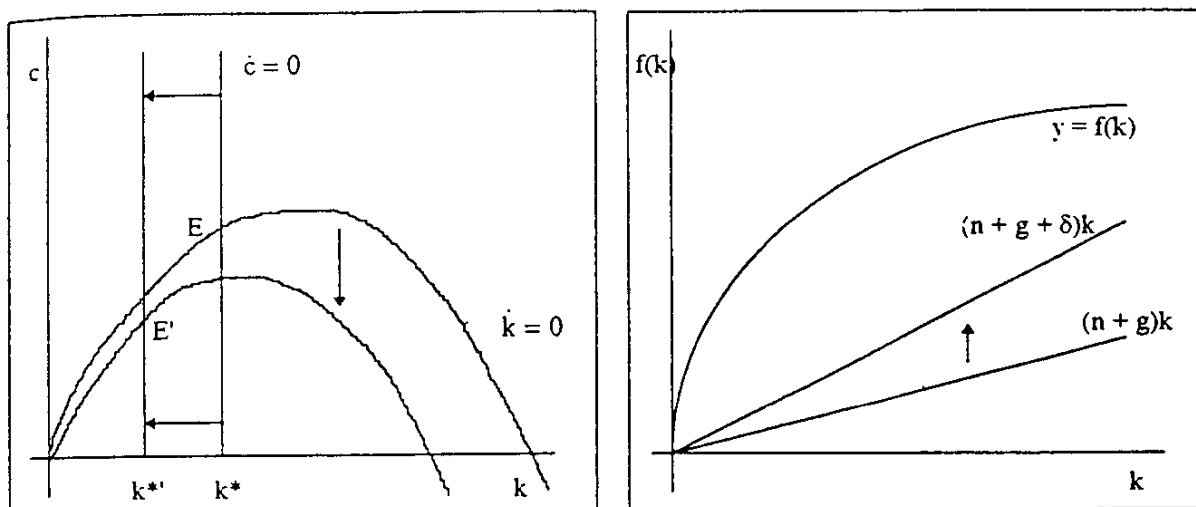
b) A downward shift of the production function.

We can assume that this means for any given k , both $f(k)$ and $f'(k)$ are lower than before.



The condition required for $\dot{k} = 0$ is given by $c = f(k) - (n + g)k$. We can see from the right diagram that the $\dot{k} = 0$ locus will shift down more at higher levels of k . Also, since for a given k , $f'(k)$ is lower now, the golden-rule k will be lower than before. Thus the $\dot{k} = 0$ locus shifts as depicted in the diagram. The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. For a given k , $f'(k)$ is now lower. Thus we need a lower k to keep $f'(k)$ the same and satisfy the $\dot{c} = 0$ equation. Thus the $\dot{c} = 0$ locus shifts left. The economy will eventually reach point E' with lower c^* and lower k^* . Whether c initially jumps up or down depends upon whether the new saddle path passes above or below point E.

c) A positive rate of depreciation.



The new capital-accumulation equation is:

$$(3) \quad \dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$$

The level of saving and investment required just to keep any given k constant is now higher -- and thus the amount of consumption possible is now lower -- than in the case with no depreciation. The level of extra investment required is also higher at higher levels of k . Thus the $\dot{k} = 0$ locus shifts down more at higher levels of k .

In addition, the real return on capital is now $f'(k(t)) - \delta$ and so the household's maximization will yield:

$$(4) \quad \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta}$$

The condition required for $\dot{c} = 0$ is $f'(k) = \delta + \rho + \theta g$. Compared to the case with no depreciation, $f'(k)$ must be higher and k lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left. The economy will eventually wind up at point E' with lower levels of c^* and k^* . Again, whether $\dot{c}$ jumps up or down initially depends upon whether the new saddle path passes above or below the original equilibrium point of E .

Problem 2.7

With a positive depreciation rate -- $\delta > 0$ -- the Euler equation and the capital-accumulation equation are given by:

$$(1) \quad \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} \quad \text{and} \quad (2) \quad \dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t)$$

We begin by taking first-order Taylor approximations to (1) and (2) around $k = k^*$ and $c = c^*$. That is, we can write:

$$(3) \quad \dot{c} \cong \frac{\partial \dot{c}}{\partial k} [k - k^*] + \frac{\partial \dot{c}}{\partial c} [c - c^*] \quad \text{and} \quad (4) \quad \dot{k} \cong \frac{\partial \dot{k}}{\partial k} [k - k^*] + \frac{\partial \dot{k}}{\partial c} [c - c^*]$$

where $\partial \dot{c} / \partial k$, $\partial \dot{c} / \partial c$, $\partial \dot{k} / \partial k$ and $\partial \dot{k} / \partial c$ are all evaluated at $k = k^*$ and $c = c^*$.

Since c^* and k^* are constants, $\dot{c}$ and $\dot{k}$ are equivalent to $c - c^*$ and $k - k^*$ respectively. We can therefore rewrite (3) and (4) as:

$$(5) \quad c - c^* \equiv \frac{\partial c}{\partial k} [k - k^*] + \frac{\partial c}{\partial c} [c - c^*] \quad \text{and} \quad (6) \quad k - k^* \equiv \frac{\partial k}{\partial k} [k - k^*] + \frac{\partial k}{\partial c} [c - c^*]$$

Using equations (1) and (2) to compute these derivatives yields:

$$(7) \quad \left. \frac{\partial c}{\partial k} \right|_{\text{bgp}} = \frac{f''(k^*)c^*}{\theta} \quad (8) \quad \left. \frac{\partial c}{\partial c} \right|_{\text{bgp}} = \frac{f'(k^*) - \delta - \rho - \theta g}{\theta} = 0$$

$$(9) \quad \left. \frac{\partial k}{\partial k} \right|_{\text{bgp}} = f'(k^*) - (n + g + \delta) \quad (10) \quad \left. \frac{\partial k}{\partial c} \right|_{\text{bgp}} = -1$$

Substituting equations (7) and (8) into (5) and equations (9) and (10) into (6) yields:

$$(11) \quad c - c^* \equiv \frac{f''(k^*)c^*}{\theta} [k - k^*]$$

$$(12) \quad k - k^* \equiv [f'(k^*) - (n + g + \delta)] \cdot [k - k^*] - [c - c^*] \\ \equiv [(\delta + \rho + \theta g) - (n + g + \delta)] \cdot [k - k^*] - [c - c^*] \\ \equiv \beta [k - k^*] - [c - c^*]$$

The second line of equation (12) uses the fact that (1) implies that $f'(k^*) = \delta + \rho + \theta g$. The third line uses the definition of β as $\rho - n - (1 - \theta)g$.

Dividing equation (11) by $(c - c^*)$ and dividing equation (12) by $(k - k^*)$ yields:

$$(13) \quad \frac{c - c^*}{c - c^*} \equiv \frac{f''(k^*)c^*}{\theta} \cdot \frac{k - k^*}{c - c^*} \quad \text{and} \quad (14) \quad \frac{k - k^*}{k - k^*} \equiv \beta - \frac{c - c^*}{k - k^*}$$

Note that these are exactly the same as equations (2.30) and (2.31) in the text — adding a positive depreciation rate does not alter the expressions for the growth rates of $c - c^*$ and $k - k^*$. Thus equation (35) — the expression for μ , the constant growth rate of both $c - c^*$ and $k - k^*$ as the economy moves towards the balanced growth path — is still valid. Thus:

$$(15) \quad \mu_1 = \frac{\beta - \sqrt{\beta^2 - 4f''(k^*)c^*/\theta}}{2}$$

where we have chosen the negative growth rate so that c and k are moving towards c^* and k^* , not away from them.

Now consider the Cobb-Douglas production function — $f(k) = k^\alpha$. Thus:

$$(16) \quad f'(k^*) = \alpha k^{*\alpha-1} \equiv r^* + \delta \quad \text{and} \quad (17) \quad f''(k^*) = \alpha(\alpha - 1)k^{*\alpha-2}$$

From equation (16), squaring both sides:

$$(18) \quad (r^* + \delta)^2 = \alpha^2 k^{*\alpha-2}$$

and so equation (17) can be rewritten as:

$$(19) \quad f''(k^*) = \frac{(r^* + \delta)^2 (\alpha - 1)}{\alpha k^*} = \frac{\alpha - 1}{\alpha} \cdot \frac{(r^* + \delta)^2}{f(k^*)}$$

In addition, defining s^* to be the saving rate on the balanced growth path, we can write the balanced-growth-path level of consumption as:

$$(20) \quad c^* = (1 - s^*)f(k^*)$$

Substituting equations (19) and (20) into (15) yields:

$$\mu_1 = \frac{\beta - \sqrt{\beta^2 - 4\left(\frac{\alpha-1}{\alpha}\right) \cdot \frac{(r^*+\delta)^2}{f(k^*)\theta} \cdot (1-s^*)f(k^*)}}{2}$$

Canceling the $f(k^*)$ and multiplying through by the minus sign yields:

$$(21) \mu_1 = \frac{\beta - \sqrt{\beta^2 + 4\left(\frac{1-\alpha}{\alpha}\right) \cdot (r^*+\delta)^2 \cdot (1-s^*)}}{2}$$

On the balanced growth path, the condition required for $\dot{c} = 0$ is given by $r^* = \rho + \theta g$ and thus:

$$(22) r^* + \delta = \rho + \theta g + \delta$$

In addition, actual saving $- s^*f(k^*)$ equals break-even investment $-(n+g+\delta)k^*$ and thus:

$$(23) s^* = \frac{(n+g+\delta)k^*}{f(k^*)} = \frac{(n+g+\delta)}{k^{*\alpha-1}} = \frac{\alpha(n+g+\delta)}{(r^*+\delta)}$$

where the last step uses equation (16) $- r^* + \delta = \alpha k^{*\alpha-1}$. From equation (23), we can write:

$$(24) (1-s^*) = \frac{(r^*+\delta) - \alpha(n+g+\delta)}{(r^*+\delta)}$$

Substituting equations (22) and (24) into equation (21) yields:

$$(25) \mu_1 = \frac{\beta - \sqrt{\beta^2 + 4\left(\frac{1-\alpha}{\alpha}\right) \cdot (\rho + \theta g + \delta) \cdot [\rho + \theta g + \delta - \alpha(n+g+\delta)]}}{2}$$

Equation (25) is analogous to equation (2.37) in the text. It expresses the rate of adjustment in terms of the underlying parameters of the model. Keeping the values in the text $- \alpha = 1/3, \rho = 4\%, n = 2\%, g = 1\%$ and $\theta = 1$ and using $\delta = 3\%$ yields $\mu_1 \cong -8.8\%$. This is faster convergence than the -5.4% obtained with no depreciation.

Problem 2.8

a) The real after-tax rate of return on capital is now given by $(1-\tau) \cdot f'(k(t))$. Thus the household's maximization would now yield the following expression describing the dynamics of consumption per unit of effective labor:

$$(1) \frac{\dot{c}(t)}{c(t)} = \frac{[(1-\tau)f'(k(t)) - \rho - \theta g]}{\theta}$$

The condition required for $\dot{c} = 0$ is given by $(1-\tau)f'(k) = \rho + \theta g$. The after-tax rate of return must equal $\rho + \theta g$. Compared to the case without a tax on capital, $f'(k)$ the pre-tax rate of return on capital must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus the $\dot{c} = 0$ locus shifts to the left.

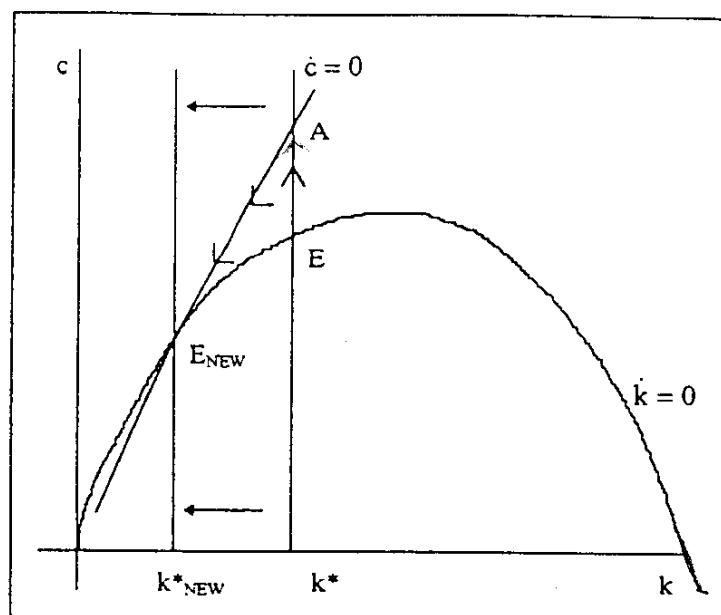
The equation describing the dynamics of the capital stock per unit of effective labor is still given by:

$$(2) \dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

For a given k , the level of c that implies $\dot{k} = 0$ is given by $c(t) = f(k) - (n+g)k$. Since the tax is rebated to households in the form of lump-sum transfers, this $\dot{k} = 0$ locus is unaffected.

b) At time 0, when the tax is put in place, the value of k — the stock of capital per unit of effective labor — is given by the history of the economy, and it cannot change discontinuously. It remains equal to the k^* on the old balanced growth path.

In contrast, c — the rate at which households are consuming in units of effective labor — can jump at the time that the tax is introduced. This jump in c is not inconsistent with the consumption-smoothing behavior implied by the household's optimization problem since the tax was unexpected and could not be prepared for.



In order for the economy to reach the new balanced growth path, it should be clear what must occur. At time 0, c jumps up so that the economy is on the new saddle path. In the diagram, the economy jumps from point E to a point like A . Basically what has happened is that since the return to saving and accumulating capital is now lower than before, people switch away from saving and into consumption.

After time 0, the economy will gradually move down the new saddle path until it eventually reaches the new balanced growth path at E_{NEW} .

c) On the new balanced growth path at E_{NEW} , the distortionary tax on investment income has caused the economy to have a lower level of capital per unit of effective labor as well as a lower level of consumption per unit of effective labor.

d) i) From the analysis above, we know that the higher is the tax rate on investment income, τ , the lower will be the balanced-growth-path level of k^* , all else equal. In terms of the above story, the higher is τ the more that the $\dot{c} = 0$ locus shifts to the left and hence the more that k^* falls. Thus $\partial k^* / \partial \tau < 0$.

On a balanced growth path, the fraction of output that is saved and invested — the saving rate — is given by $[f(k^*) - c^*] / f(k^*)$. Since k is constant, or $\dot{k} = 0$, on a balanced growth path then — from $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$ — we can write $f(k^*) - c^* = (n + g)k^*$. Using this we can rewrite the fraction of output that is saved on a balanced growth path as:

$$(3) \quad s = [(n + g)k^*] / f(k^*)$$

Use equation (3) to take the derivative of the saving rate with respect to the tax rate, τ :

$$(4) \quad \frac{\partial s}{\partial \tau} = \frac{(n + g) \cdot (\partial k^* / \partial \tau) \cdot f(k^*) - (n + g)k^* f'(k^*) \cdot (\partial k^* / \partial \tau)}{f(k^*)^2}$$

simplifying yields:

$$\frac{\partial s}{\partial \tau} = \frac{(n + g)}{f(k^*)} \cdot \frac{\partial k^*}{\partial \tau} - \frac{(n + g)}{f(k^*)} \cdot \frac{k^* f'(k^*)}{f(k^*)} \cdot \frac{\partial k^*}{\partial \tau} = \frac{(n + g)}{f(k^*)} \cdot \frac{\partial k^*}{\partial \tau} \cdot \left[1 - \frac{k^* f'(k^*)}{f(k^*)} \right]$$

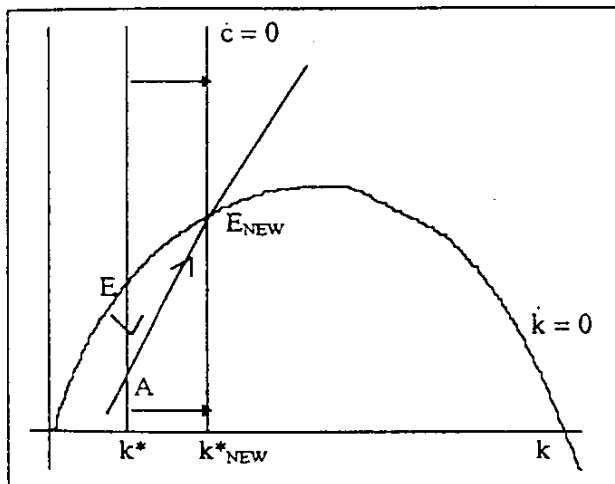
Recall that $k^* f'(k^*) / f(k^*) \equiv \alpha_K(k^*)$ is capital's (pre-tax) share in income, which must be less than one. Since $\partial k^* / \partial \tau < 0$, then we can write:

$$(5) \frac{\partial s}{\partial \tau} = \frac{(n+g)}{f(k^*)} \cdot \frac{\partial k^*}{\partial \tau} \cdot [1 - \alpha_K(k^*)] < 0$$

Thus the saving rate on the balanced growth path is decreasing in τ .

d) ii) Citizens in low- τ , high- k^* , high-saving countries do not have the incentive to invest in low-saving countries. From part (a), the condition required for $\dot{c} = 0$ is $(1 - \tau)f'(k) = \rho + \theta g$. That is, the after-tax rate of return must equal $\rho + \theta g$. Assuming preferences and technology are the same across countries -- so that ρ , θ and g are the same across countries -- the after-tax rate of return will be the same across countries. Since the after-tax rate of return is thus the same in low-saving countries as it is in high-saving countries, there is no incentive to shift saving from a high-saving to a low-saving country.

e) Should the government subsidize investment instead and fund this with a lump-sum tax? This would lead to the opposite result from above and the economy would have higher c and k on the new balanced growth path.



The answer is no. The original market outcome is already the one that would be chosen by a central planner attempting to maximize the lifetime utility of a representative household subject to the capital-accumulation equation. It therefore gives the household the highest possible lifetime utility.

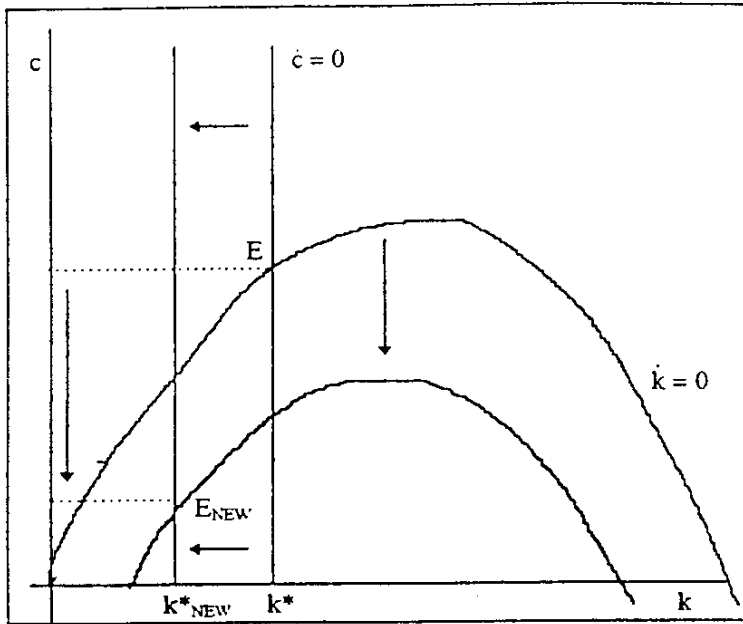
Starting at point E, the implementation of the subsidy would lead to a short-term drop in consumption at point A, but would eventually result in permanently higher consumption at point E_{NEW} . It would turn out that the utility lost from the short-term sacrifice would outweigh the utility gained in the long-term (all in present value terms, appropriately discounted).

This is the same type of argument used to explain why households do not choose to consume at the golden-rule level. See Section 2.4 for a more complete description of the welfare implications of this model.

f) Suppose the government does not rebate the tax revenue to households but instead uses it to make government purchases. Let $G(t)$ represent government purchases per unit of effective labor. The equation describing the dynamics of the capital stock per unit of effective labor is now given by:

$$(2') \quad \dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t)$$

The fact that the government is making purchases that do not add to the capital stock -- it is assumed to be government consumption not government investment -- shifts down the $\dot{k} = 0$ locus.



After the imposition of the tax, the $\dot{c} = 0$ locus shifts to the left, just as it did in the case where the government rebated the tax to households. In the end, k^* falls to k^*_{NEW} just as in the case where the government rebated the tax. Consumption per unit of effective labor on the new balanced growth path at E_{NEW} is lower -- compared to the case where the tax is rebated -- by the amount of the government purchases, which is $\tau f'(k)k$.

Finally, whether the level of c jumps up or down initially depends upon whether the new saddle path passes above or below the original balanced-growth-path point of E .

Problem 2.9

a) - c) Before the tax is put in place -- until time t_1 -- the equations governing the dynamics of the economy are:

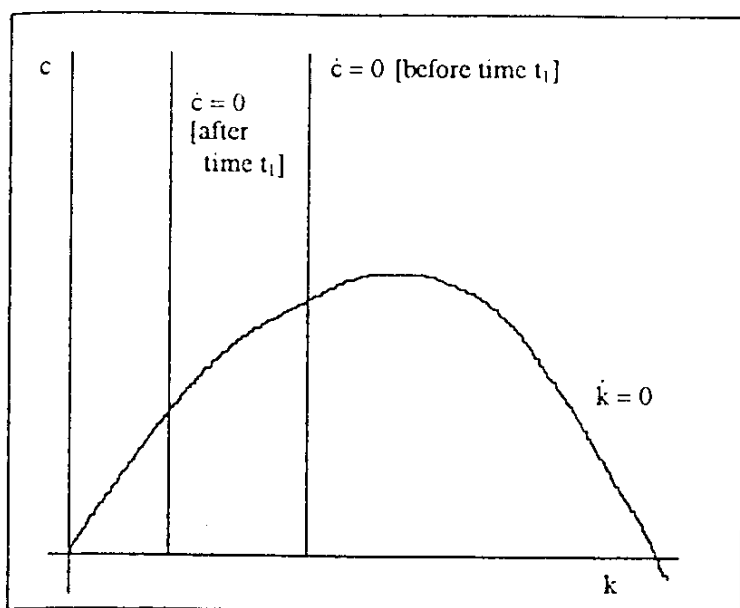
$$(1) \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta} \quad \text{and} \quad (2) \dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$$

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation is not affected when the tax is put in place at time t_1 since we are assuming that the government is rebating the tax, not spending it.

Since the real after-tax rate of return on capital is now $(1 - \tau)f'(k(t))$, the household's maximization yields the following growth rate of consumption:

$$(3) \frac{\dot{c}(t)}{c(t)} = \frac{(1 - \tau)f'(k(t)) - \rho - \theta g}{\theta}$$

The condition required for $\dot{c} = 0$ is now given by $(1 - \tau)f'(k) = \rho + \theta g$. The after-tax rate of return on capital must equal $\rho + \theta g$. Thus the pre-tax rate of return, $f'(k)$, must be higher and thus k must be lower in order for $\dot{c} = 0$. Thus at time t_1 , the $\dot{c} = 0$ locus shifts to the left.

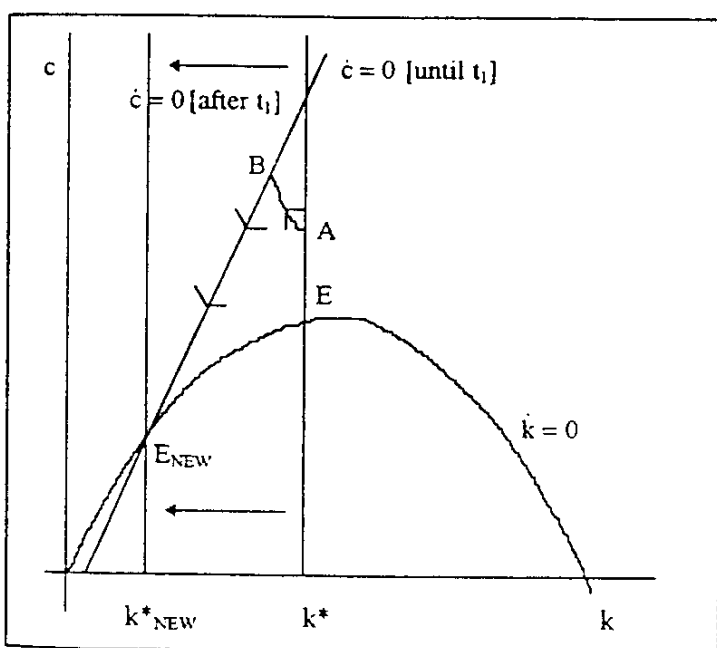


The important point is that the dynamics of the economy are still governed by the original equations of motion until the tax is actually put in place. Between the time of the announcement and the time the tax is actually put in place, it is the original $\dot{c} = 0$ locus that is relevant.

When the tax is put in place at time t_1 , c cannot jump discontinuously because everyone knows ahead of time that the tax will be implemented then. A discontinuous jump in c would be inconsistent with the consumption smoothing implied by the household's intertemporal optimization. The household would not want c to be low -- and thus marginal utility to be high -- today, knowing that c will jump up and be

high -- and thus marginal utility will be low -- tomorrow. The household would like to smooth consumption between the two instants in time.

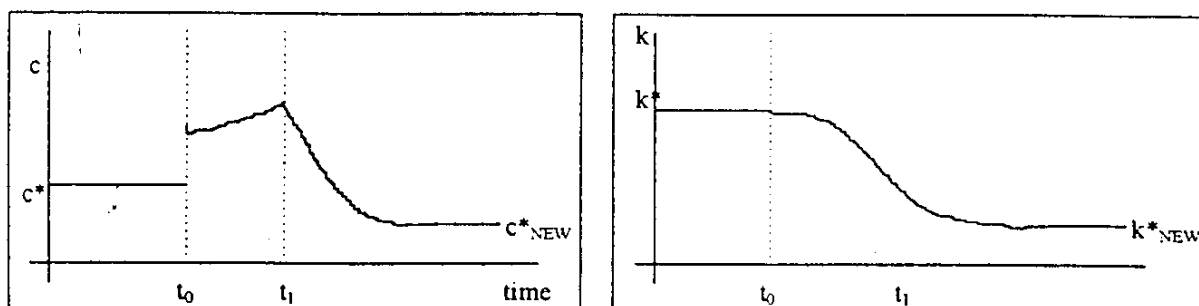
d) We know that c cannot jump at time t_1 . We also know that if the economy is to reach the new balanced growth path at point E_{NEW} , it must be right on the new saddle path at the time that the tax is put in place. Thus when the tax is announced at time t_0 , c must jump up to a point like A . Point A lies between the original balanced growth path at E and the new saddle path.



At A , c is too high to maintain the capital stock at k^* and so k begins falling. Between t_0 and t_1 , the dynamics of the system are still governed by the original $\dot{c} = 0$ locus. The economy is thus to the left of the $\dot{c} = 0$ locus and so consumption begins rising.

The economy moves off to the northwest until at t_1 , it is right at point B on the new saddle path. The tax is then put in place and the system is governed by the new $\dot{c} = 0$ locus. Thus c begins falling. The economy moves down the new saddle path, eventually reaching point E_{NEW} .

e) The story in part d implies the following time paths for consumption per unit of effective labor and capital per unit of effective labor.



Problem 2.10

a) The first point is that consumption cannot jump at time t_1 . Everyone knows ahead of time that the tax will end then and so a discontinuous jump in c would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. Thus for the economy to return to a balanced growth path, we must be somewhere on the original saddle path right at time t_1 .

Before the tax is put in place — until time t_0 — and after the tax is removed — after time t_1 — the equations governing the dynamics of the economy are:

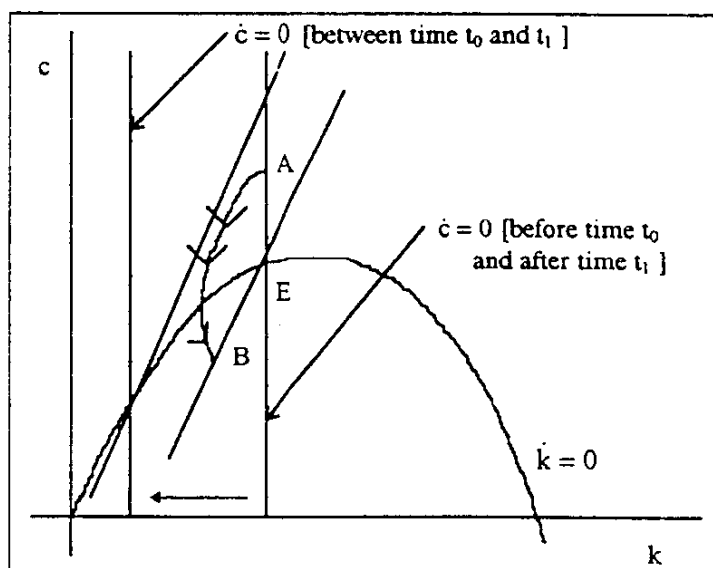
$$(1) \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta} \quad \text{and} \quad (2) \dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$$

The condition required for $\dot{c} = 0$ is given by $f'(k) = \rho + \theta g$. The capital-accumulation equation — and thus the $\dot{k} = 0$ locus — is not affected by the tax. The $\dot{c} = 0$ locus is affected, however. Between time t_0 and time t_1 , the condition required for $\dot{c} = 0$ is that the after-tax rate of return on capital equal $\rho + \theta g$ — that is $(1 - \tau)f'(k) = \rho + \theta g$. Thus, between t_0 and t_1 , $f'(k)$ must be higher and so k must be lower in order for $\dot{c} = 0$. That is, between time t_0 and time t_1 , the $\dot{c} = 0$ locus lies to the left of its original position.

At time t_0 , the tax is put in place. At point E, the economy is still on the $\dot{k} = 0$ locus but is now to the right of the new $\dot{c} = 0$ locus. Thus if c did not jump up to a point like A, c would begin falling. The economy would then be below the $\dot{k} = 0$ locus and so k would start rising. The economy would drift away from point E in the direction of the southeast and could not be on the original saddle path right at time t_1 .

Thus at time t_0 , c must jump up so that the economy is at a point like A. Thus k and c begin falling. Eventually the economy crosses the $\dot{k} = 0$ locus and so

k begins rising. This can be loosely interpreted as people anticipating the removal of the tax on capital and thus being willing to accumulate capital again. Point A must be such that given the dynamics of the system, the economy is right at a point like B — on the original saddle path — at time t_1 when the tax is



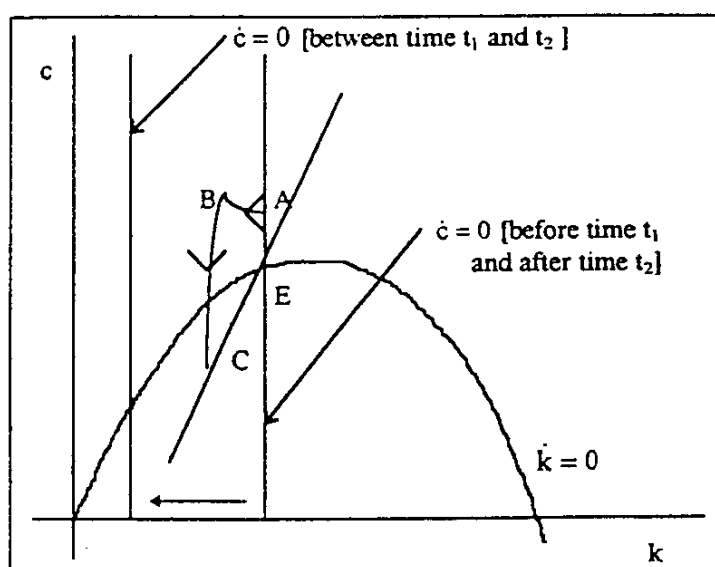
removed. After t_1 , the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

b) The first point is that consumption cannot jump at either time t_1 or time t_2 . Everyone knows ahead of time that the tax will be implemented at t_1 and removed at t_2 . Thus a discontinuous jump in c at either date would be inconsistent with the consumption-smoothing behavior implied by the household's intertemporal optimization. Thus for the economy to return to a balanced growth path, the economy must be somewhere on the original saddle path right at time t_2 .

Before the tax is put in place — until time t_1 — and after the tax is removed — after time t_2 — equations (1) and (2) govern the dynamics of the system. An important point is that even during the time between the announcement and the implementation of the tax — that is, between time t_0 and time t_1 — the original $\dot{c} = 0$ locus governs the dynamics of the system.

At time t_0 , the tax is announced. Consumption must jump up so that the economy is at a point like A. At A, the economy is still on the $\dot{c} = 0$ locus but is above the $\dot{k} = 0$ locus and so k starts falling. The economy is then to the left of the $\dot{c} = 0$ locus and so c starts rising. The economy drifts off to the northwest.

At time t_1 , the tax is implemented, the $\dot{c} = 0$ locus shifts to the left and the economy is at a point like B. The economy is still above the $\dot{k} = 0$ locus but is now to the right of the relevant $\dot{c} = 0$ locus — k continues to fall and c stops rising and begins to fall.



Eventually the economy crosses the $\dot{k} = 0$ locus and k begins rising. Again, people begin accumulating capital again before the actual removal of the tax on capital income. Point A must be chosen so that given the dynamics of the system, the economy is right at a point like C — on the original saddle path — at time t_2 when the tax is removed. After t_2 , the original $\dot{c} = 0$ locus governs the dynamics of the system again. Thus the economy moves up the original saddle path, eventually returning to the original balanced growth path at point E.

Problem 2.11

With government purchases in the model, the capital-accumulation equation is given by:

$$(1) \quad \dot{k}(t) = f(k(t)) - c(t) - G(t) - (n + g)k(t).$$

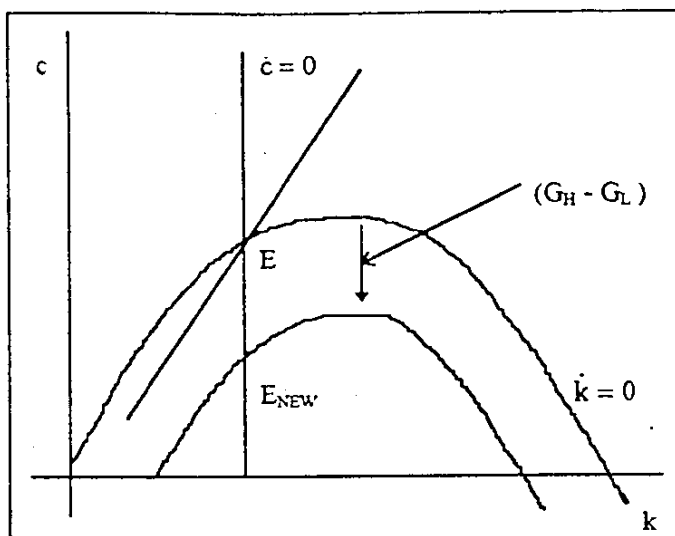
where $G(t)$ represents government purchases in units of effective labor at time t .

Intuitively, since government purchases are assumed to be a perfect substitute for private consumption, changes in G will simply be offset one-for-one with changes in c . Suppose that $G(t)$ is initially constant at some level G_L . The household's maximization yields:

$$(2) \frac{\dot{c}(t)}{c(t) + G_L} = \frac{f'(k(t)) - \rho - \theta g}{\theta}$$

Thus the condition for constant consumption is still given by $f'(k) = \rho + \theta g$. Changes in the level of G_L will affect the level of c , but will not shift the $\dot{c} = 0$ locus.

Suppose the economy starts on a balanced growth path at point E. At some time t_0 , G unexpectedly increases to G_H and everyone knows this is temporary — everyone knows that at some future time t_1 , government purchases will return to G_L . At time t_0 , the $\dot{k} = 0$ locus shifts down — at each level of k , the government is using more resources leaving less available for consumption. In particular, the $\dot{k} = 0$ locus shifts down by the amount of the increase in purchases, which is $(G_H - G_L)$.



The difference between this case — where c and G are perfect substitutes — and the case where G does not affect private utility, is that c can jump at time t_1 when G returns to its original value. In fact, at t_1 , when G jumps down by the amount $(G_H - G_L)$, c must jump up by that exact same amount. If it did not, there would be a discontinuous jump in marginal utility that could not be optimal for households. Thus at t_1 , c must jump up by $(G_H - G_L)$ and this must put the economy somewhere on the original saddle path. If it did not, the economy would not return to a balanced growth path. What must happen is that at time t_0 , c falls by the amount $(G_H - G_L)$ and the economy jumps to point E_{NEW} . It then stays there until time t_1 . At t_1 , c jumps back up by the amount $(G_H - G_L)$ and so the economy jumps back to point E and stays there.

Why can't c jump down by less than $(G_H - G_L)$ at t_0 ? If it did, the economy would be above the new $\dot{k} = 0$ locus, k would start falling putting the economy to the left of the $\dot{c} = 0$ locus. Thus c would start rising and so the economy would drift off to the northwest. There would be no way for c to jump up by $(G_H - G_L)$ at t_1 and still put the economy on the original saddle path.

Why can't c jump down by more than $(G_H - G_L)$ at t_0 ? If it did, the economy would be below the new $\dot{k} = 0$ locus, k would start rising putting the economy to the right of the $\dot{c} = 0$ locus. Thus c would start falling and so the economy would drift off to the southeast. Again, there would be no way for c to jump up by $(G_H - G_L)$ at t_1 and still put the economy on the original saddle path.

In summary, the capital stock and the real interest rate are unaffected by the temporary increase in G . At the instant that G rises, consumption falls by an equal amount. It remains constant at that level while G remains high. At the instant that G falls to its initial value, consumption jumps back up to its original value and stays there.

Problem 2.12

Throughout, we will assume $U'(\bullet) > 0$ and $U''(\bullet) < 0$. In addition, since the expected value of Y_2 is equal to Y_1 , we can write $Y_2 = Y_1 + \epsilon$ with $E(\epsilon) = 0$.

a) The individual's problem is to choose C_1 and C_2 in order to maximize $U(C_1) + E[U(C_2)]$ subject to:

$$(1) C_2 = (1 - \tau_1)Y_1 - C_1 + (1 - \tau_2)(Y_1 + \varepsilon)$$

We can substitute for C_2 and solve the unconstrained problem of choosing C_1 :

$$\max U(C_1) + E[U\{(1 - \tau_1)Y_1 - C_1 + (1 - \tau_2)(Y_1 + \varepsilon)\}]$$

The first-order condition is given by:

$$U'(C_1) + E[U'(C_2)(-1)] = 0$$

or simply:

$$(2) U'(C_1) = E[U'(C_2)]$$

If the individual is optimizing, the marginal utility of consumption in period 1 must equal the expected marginal utility of consumption in period 2.

b) If Y_2 is not random, the first-order condition reduces to $U'(C_1) = U'(C_2)$. With $U''(\bullet) < 0$ everywhere, this implies that $C_1 = C_2$. If utility is quadratic then $U'(C)$ is linear in C . Using the hint --

$E[U'(C_2)] = U'[E(C_2)]$ -- the first-order condition given by equation (2) can be rewritten as

$$U'(C_1) = U'[E(C_2)]. \text{ Since } U''(\bullet) < 0 \text{ everywhere, this implies that } C_1 = E(C_2).$$

c) Now, $U'(\bullet) > 0$, $U''(\bullet) < 0$ and $U'''(\bullet) > 0$.

Marginal utility is now a convex function of consumption and so by Jensen's inequality,

$$E[U'(C_2)] > U'[E(C_2)]. \text{ Combining this with the first-order condition -- } U'(C_1) = E[U'(C_2)] \text{ -- yields}$$

$$U'(C_1) > U'[E(C_2)]. \text{ Since } U'(\bullet) > 0 \text{ and } U''(\bullet) < 0 \text{ we have } C_1 < E(C_2). \text{ The individual plans in such}$$

a way that if second-period income turns out to be equal to its expected value, C_2 would turn out to be higher than C_1 . Thus, in the face of uncertainty and with $U'''(\bullet) > 0$, the individual undertakes "precautionary saving".

d) The government is cutting first-period taxes (τ_1) and raising second-period taxes (τ_2) in such a way that expected tax revenue remains unchanged. Expected tax revenue, $\bar{R}$, can be expressed as

$$\tau_1 Y_1 + \tau_2 E(Y_1 + \varepsilon) = \bar{R}. \text{ Using the fact that } E(\varepsilon) = 0, \text{ we can solve for } \tau_2:$$

$$\tau_1 Y_1 + \tau_2 Y_1 = \bar{R} \Rightarrow \tau_2 = \bar{R}/Y_1 - \tau_1$$

In order to keep $\bar{R}$ constant, the change in taxes must satisfy:

$$(3) \partial \tau_2 / \partial \tau_1 = -1$$

The question is whether or not this change in the timing of taxes alters the individual's consumption behavior. Substitute equation (1) into the first-order condition -- equation (2) -- to obtain:

$$(4) U'(C_1) = E[U'\{(1 - \tau_1)Y_1 - C_1 + (1 - \tau_2)(Y_1 + \varepsilon)\}]$$

Differentiating both sides of equation (4) with respect to τ_1 yields:

$$U''(C_1) \cdot \partial C_1 / \partial \tau_1 = E[U''(C_2)\{-Y_1 - \partial C_1 / \partial \tau_1 - (Y_1 + \varepsilon) \cdot \partial \tau_2 / \partial \tau_1\}]$$

and now using equation (3) -- $\partial \tau_2 / \partial \tau_1 = -1$ -- we have

$$U''(C_1) \cdot \partial C_1 / \partial \tau_1 = E[U''(C_2)\{-Y_1 - \partial C_1 / \partial \tau_1 + Y_1 + \varepsilon\}]$$

$$U''(C_1) \cdot \partial C_1 / \partial \tau_1 = E[U''(C_2) \cdot (-\partial C_1 / \partial \tau_1)] + E[U''(C_2) \cdot \varepsilon]$$

or:

$$[U''(C_1) + E[U''(C_2)]] \cdot \partial C_1 / \partial \tau_1 = E[U''(C_2) \cdot \varepsilon]$$

Now use the fact that for any two random variables X and Y , $E(XY) = E(X)E(Y) + \text{cov}(X, Y)$:

$$[U'(C_1) + E[U'(C_2)]] \cdot \partial C_1 / \partial \tau_1 = E[U'(C_2)] \cdot E[\varepsilon] + \text{cov}[U'(C_2), \varepsilon]$$

Finally, $E(\varepsilon) = 0$ and thus the change in first-period consumption due to this change in the timing of taxes is given by:

$$(5) \quad \frac{\partial C_1}{\partial \tau_1} = \frac{\text{cov}[U'(C_2), \varepsilon]}{U'(C_1) + E[U'(C_2)]}$$

e) If Y_2 is not random then $\varepsilon = 0$ always and thus the covariance in the numerator of equation (5) is 0. In addition, if utility is quadratic, then $U''(\bullet)$ is a constant and again the covariance is 0. In both of these cases $\partial C_1 / \partial \tau_1 = 0$ and thus first-period consumption does not change in response to the tax cut.

f) In the case of $U'''(\bullet) > 0$ we need to show that $\partial C_1 / \partial \tau_1 < 0$. That is, we need to show that C_1 rises in response to the reduction in τ_1 . Intuitively, the higher is ε , the higher will be C_2 . The individual simply consumes any extra random income in the second period. If $U'''(\bullet) > 0$, then as C_2 rises so will $U''(C_2)$, and thus it will be the case that $\text{cov}[U''(C_2), \varepsilon] > 0$. The denominator of equation (5) will be negative since $U''(\bullet) < 0$ and thus $\partial C_1 / \partial \tau_1 < 0$ as required. The intuition is that the change in the timing of taxes leaves the individual with the same expected after-tax lifetime income, but more of it comes with certainty in the first period. If the individual is doing precautionary saving – if $U'''(\bullet) > 0$ she is – the amount of such saving will be reduced and she will consume more in the first period.

g) The CRRA utility function and its first two derivatives are given by:

$$U(C) = C^{1-\theta} / (1-\theta) \Rightarrow U'(C) = C^{-\theta} \Rightarrow U''(C) = -\theta C^{-\theta-1}$$

and thus:

$$U''(C) = -\theta(-\theta-1)C^{-\theta-2} = (\theta^2 + \theta)C^{-\theta-2} > 0 \quad \text{for } \theta > 0$$

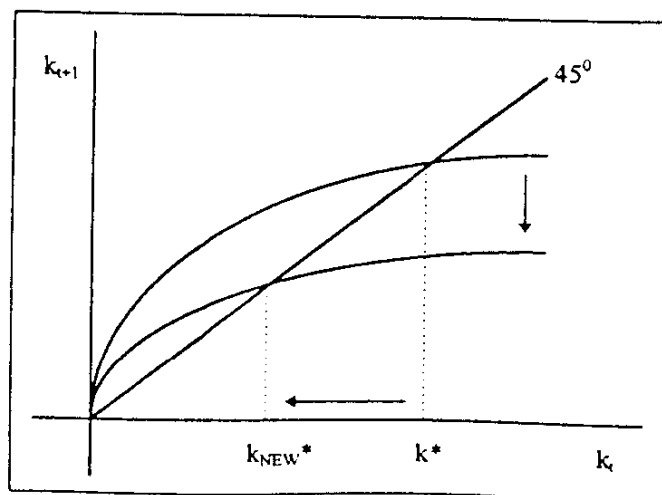
Thus the CRRA utility function does imply the precautionary saving behavior described above.

Problem 2.13

Equation (2.61) in the text describes the relationship between k_{t+1} and k_t in the special case of logarithmic utility and Cobb-Douglas production:

$$(2.61) \quad k_{t+1} = \frac{1}{(1+n)(1+g)} \cdot \frac{1}{2+\rho} \cdot (1-\alpha)k_t^\alpha \equiv Dk_t^\alpha$$

a) A rise in n shifts the k_{t+1} function down. From equation (2.61), a higher n means a smaller D and thus a smaller k_{t+1} for a given k_t . Since the fraction of their labor income that the young save does not depend on n , a given amount of capital per unit of effective labor and thus output per unit of effective labor in time t yields the same amount of saving in period t . Thus it yields the same amount of capital in period $t+1$. However, the number of people increases more from period t to period $t+1$ than it used to. So that capital is spread out among more people



than it would have been in the absence of the increase in population growth and thus capital per unit of effective labor in period $t + 1$ is lower for a given k_t .

b) With the parameter B added to the Cobb-Douglas production function — $f(k) = Bk^\alpha$ — equation (2.61) becomes:

$$(1) \quad k_{t+1} = \frac{1}{(1+n)(1+g)} \cdot \frac{1}{2+\rho} \cdot (1-\alpha)Bk_t^\alpha$$

This fall in B causes the k_{t+1} function to shift down. See the diagram from part a. A lower B means that a given amount of capital per unit of effective labor in period t now produces less output per unit of effective labor in period t . Since the fraction of their labor income that the young save does not depend on B , this leads to less total saving and a lower capital stock per unit of effective labor in period $t + 1$ for a given k_t .

c) We need to determine the effect on k_{t+1} for a given k_t , of a change in α . From equation (2.61):

$$(2) \quad \frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)} \cdot \frac{1}{2+\rho} \cdot \left[-k_t^\alpha + (1-\alpha) \frac{\partial k_t^\alpha}{\partial \alpha} \right]$$

We need to determine $\partial k_t^\alpha / \partial \alpha$. Define $f(\alpha) \equiv k_t^\alpha$ and note that $\ln f(\alpha) = \alpha \ln k_t$. Thus:

$$(3) \quad \partial \ln f(\alpha) / \partial \alpha = \ln k_t$$

Now note that we can write:

$$(4) \quad \frac{\partial f(\alpha)}{\partial \alpha} = \frac{\partial f(\alpha)}{\partial \ln f(\alpha)} \frac{\partial \ln f(\alpha)}{\partial \alpha} = \frac{1}{[\partial \ln f(\alpha) / \partial f(\alpha)]} \frac{\partial \ln f(\alpha)}{\partial \alpha}$$

and thus finally:

$$(5) \quad \partial f(\alpha) / \partial \alpha = f(\alpha) \ln k_t$$

Therefore, we have $\partial k_t^\alpha / \partial \alpha = k_t^\alpha \ln k_t$. Substituting this fact into equation (2) yields:

$$(6) \quad \frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)} \cdot \frac{1}{2+\rho} \cdot \left[-k_t^\alpha + (1-\alpha)k_t^\alpha \ln k_t \right]$$

or simply:

$$(7) \quad \frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)} \cdot \frac{1}{2+\rho} \cdot \left\{ k_t^\alpha [(1-\alpha) \ln k_t - 1] \right\}$$

Thus, for $(1-\alpha) \ln k_t - 1 > 0$, or $\ln k_t > 1/(1-\alpha)$, an increase in α means a higher k_{t+1} for a given k_t and thus the k_{t+1} function shifts up over this range of k_t 's. However, for $\ln k_t < 1/(1-\alpha)$, an increase in α means a lower k_{t+1} for a given k_t . Thus the k_{t+1} function shifts down over this range of k_t 's. Finally, right at $\ln k_t = 1/(1-\alpha)$, the old and new k_{t+1} functions intersect.

Problem 2.14

a) We need to find an expression for k_{t+1} as a function of k_t . Next period's capital stock is equal to this period's capital stock plus any investment done this period less any depreciation that occurs. Thus:

$$(1) \quad K_{t+1} = K_t + sY_t - \delta K_t$$

To convert this into units of effective labor, divide both sides of equation (1) by $A_{t+1}L_{t+1}$:

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}} = \frac{K_t(1-\delta) + sY_t}{A_{t+1}L_{t+1}} = \frac{K_t(1-\delta) + sY_t}{(1+n)(1+g)A_tL_t} = \frac{k_t(1-\delta) + sf(k_t)}{(1+n)(1+g)}$$

which simplifies to:

$$(2) \quad k_{t+1} = \left[\frac{1-\delta}{(1+n)(1+g)} \right] k_t + \left[\frac{s}{(1+n)(1+g)} \right] f(k_t)$$

b) We need to sketch k_{t+1} as a function of k_t :

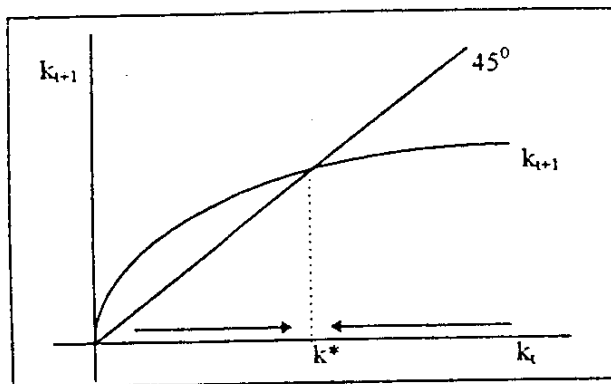
Note that:

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{1-\delta}{(1+n)(1+g)} + \left[\frac{s}{(1+n)(1+g)} \right] f'(k_t) > 0 \quad \text{and} \quad \frac{\partial^2 k_{t+1}}{\partial k_t^2} = \frac{sf''(k_t)}{(1+n)(1+g)} < 0$$

and using the Inada conditions:

$$\lim_{k \rightarrow 0} \frac{\partial k_{t+1}}{\partial k_t} = \infty \quad \lim_{k \rightarrow \infty} \frac{\partial k_{t+1}}{\partial k_t} = \frac{1-\delta}{(1+n)(1+g)} < 1$$

Thus the function eventually has a slope of less than 1 and will therefore cross the 45 degree line at some point. Also, the function is well-behaved and will cross the 45 degree line only once.



As long as k starts out at some value other than 0, the economy will converge to k^* . For example, if k started out below k^* , we see that k_{t+1} will be greater than k_t and the economy will move towards k^* . Similarly, if k starts out above k^* , we see that k_{t+1} will be below k_t and again the economy will move towards k^* . At k^* , $y^* = f(k^*)$ is also constant and we have a balanced growth path.

c) On a balanced growth path, $k_{t+1} = k_t \equiv k^*$ and thus from equation (2):

$$k^* = \left[\frac{1-\delta}{(1+n)(1+g)} \right] k^* + \left[\frac{s}{(1+n)(1+g)} \right] f(k^*)$$

which simplifies to:

$$k^* \left[\frac{1+n+g+ng-1+\delta}{(1+n)(1+g)} \right] = \left[\frac{s}{(1+n)(1+g)} \right] f(k^*)$$

Thus on a balanced growth path:

$$(3) \quad k^*(n+g+ng+\delta) = sf(k^*)$$

Rearranging equation (3) to get an expression for s on the balanced growth path yields:

$$(4) \quad s = (n+g+ng+\delta)k^*/f(k^*)$$

Consumption per unit of effective labor on the balanced growth path is given by:

$$(5) \quad c^* = (1-s)f(k^*)$$

Substitute equation (4) into equation (5):

$$c^* = \left[1 - \frac{(n+g+ng+\delta)k^*}{f(k^*)} \right] f(k^*) = \left[\frac{f(k^*) - k^*(n+g+ng+\delta)}{f(k^*)} \right] f(k^*)$$

Canceling the $f(k^*)$ yields:

$$(6) \quad c^* = f(k^*) - (n+g+ng+\delta)k^*$$

To get an expression for the $f'(k^*)$ that maximizes consumption per unit of effective labor on the balanced growth path, we need to maximize c^* with respect to k^* . The first-order condition is given by:

$$\partial c^* / \partial k^* = f'(k^*) - (n+g+ng+\delta) = 0$$

Thus the golden-rule capital stock is defined implicitly by:

$$(7) \quad f'(k_{GR}) = (n+g+ng+\delta)$$

d) i) Substitute a Cobb-Douglas production function -- $f(k_t) = k_t^\alpha$ -- into equation (2):

$$(8) \quad k_{t+1} = \left[\frac{1-\delta}{(1+n)(1+g)} \right] k_t + \left[\frac{s}{(1+n)(1+g)} \right] k_t^\alpha$$

ii) On a balanced growth path, $k_{t+1} = k_t \equiv k^*$. Thus from equation (8):

$$k^* = \left[\frac{1-\delta}{(1+n)(1+g)} \right] k^* + \left[\frac{s}{(1+n)(1+g)} \right] k^{*\alpha}$$

Simplifying yields:

$$\left[\frac{(1+n)(1+g) - (1-\delta)}{(1+n)(1+g)} \right] k^* = \left[\frac{s}{(1+n)(1+g)} \right] k^{*\alpha} \Rightarrow k^{*1-\alpha} = s/(n+g+ng+\delta)$$

and thus finally:

$$(9) \quad k^* = [s/(n+g+ng+\delta)]^{1/(1-\alpha)}$$

iii) Using equation (8):

$$(10) \quad \left. \frac{dk_{t+1}}{dk_t} \right|_{k_t=k^*} = \frac{1-\delta}{(1+n)(1+g)} + \frac{\alpha s}{(1+n)(1+g)} \cdot k^{*\alpha-1}$$

Substituting the balanced-growth-path value of k^* -- equation (9) -- into equation (10) yields:

$$\left. \frac{dk_{t+1}}{dk_t} \right|_{k_t=k^*} = \frac{1-\delta}{(1+n)(1+g)} + \frac{\alpha s}{(1+n)(1+g)} \cdot \left[\frac{(n+g+ng+\delta)}{s} \right]$$

It will be useful to write $(n+g+ng+\delta)$ as $(1+n)(1+g) - (1-\delta)$:

$$\left. \frac{dk_{t+1}}{dk_t} \right|_{k_t=k^*} = \frac{(1-\delta) + \alpha[(1+n)(1+g) - (1-\delta)]}{(1+n)(1+g)}$$

Simplifying further yields:

$$(11) \quad \left. \frac{dk_{t+1}}{dk_t} \right|_{k_t=k^*} = \alpha + \frac{(1-\delta)(1-\alpha)}{(1+n)(1+g)}$$

Replacing equation (8) by its first-order Taylor approximation around $k = k^*$ therefore gives us:

$$(12) \quad k_{t+1} \approx k^* + \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g) \right] \cdot [k_t - k^*]$$

Since we can write this simply as:

$$k_{t+1} - k^* \approx \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g) \right] \cdot [k_t - k^*]$$

equation (12) implies:

$$(13) \quad k_t - k^* \approx \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g) \right]^t \cdot [k_0 - k^*]$$

Thus the economy moves fraction $1 - \left[\alpha + (1-\delta)(1-\alpha)/(1+n)(1+g) \right]$ of the way to the balanced growth path each period. Some simple algebra simplifies the expression for this rate of convergence to:

$$(1-\alpha)(n+g+ng+\delta)/(1+n)(1+g)$$

With $\alpha = 1/3$, $n = 1\%$, $g = 2\%$ and $\delta = 3\%$, this yields a rate of convergence of $\approx 3.9\%$. This is similar but slower than the rate of convergence found in the continuous time Solow model.

Problem 2.15

a) The individual's optimization problem is not affected by the depreciation which means that $r_t = f'(k_t) - \delta$. The household's problem is still to maximize utility:

$$(1) U_t = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \cdot \frac{C_{2,t+1}^{1-\theta}}{1-\theta}$$

subject to the budget constraint:

$$(2) C_{1,t} + \frac{1}{1+r_{t+1}} \cdot C_{2,t+1} = A_t w_t$$

As in the text – with no depreciation – the fraction of income saved, $s(r_{t+1}) \equiv (1 - C_{1,t}) A_t w_t$, is given by:

$$(3) s(r_{t+1}) = \frac{1}{1 + (1+\rho)^{1/\theta} (1+r_{t+1})^{(\theta-1)/\theta}}$$

Thus the way in which the fraction of income saved depends on the real interest rate, r_{t+1} , is unchanged.

The only difference is that the real interest rate itself is now $f'(k_{t+1}) - \delta$, rather than just $f'(k_{t+1})$.

The capital stock in period $t+1$ equals the amount saved by young individuals in period t . Thus:

$$(4) K_{t+1} = S_t L_t$$

where S_t is the amount of saving done by a young person in period t . Note that $S_t \equiv s(r_{t+1}) \cdot A_t w_t$ – the amount of saving done is equal to the fraction of income saved times the amount of income. Thus equation

(4) can be rewritten as:

$$(5) K_{t+1} = L_t \cdot s(r_{t+1}) \cdot A_t w_t$$

To get this into units of time $t+1$ effective labor, divide both sides of equation (5) by $A_{t+1} L_{t+1}$:

$$(6) \frac{K_{t+1}}{A_{t+1} L_{t+1}} = \frac{A_t L_t}{A_{t+1} L_{t+1}} \cdot [s(r_{t+1}) \cdot w_t]$$

Since $A_{t+1} = (1+g)A_t$, we have $A_t/A_{t+1} = 1/(1+g)$. Similarly, $L_t/L_{t+1} = 1/(1+n)$. In addition, $K_{t+1}/A_{t+1} L_{t+1} \equiv k_{t+1}$. Thus:

$$(7) k_{t+1} = \frac{1}{(1+n)(1+g)} \cdot [s(r_{t+1}) \cdot w_t]$$

Finally, substitute for $r_{t+1} = f'(k_{t+1}) - \delta$ and $w_t = f(k_t) - k_t f'(k_t)$:

$$(8) k_{t+1} = \frac{1}{(1+n)(1+g)} \cdot [s(f'(k_{t+1}) - \delta)] \cdot [f(k_t) - k_t f'(k_t)]$$

This should be compared with equation (2.60) in the text – the analogous expression with no depreciation – which is:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \cdot [s(f'(k_{t+1}))] \cdot [f(k_t) - k_t f'(k_t)]$$

Thus adding depreciation does alter the relationship between k_{t+1} and k_t . Whether k_{t+1} will be higher or lower for a given k_t depends on how saving varies with r_{t+1} .

b) With logarithmic utility, the fraction of income saved does not depend upon the rate of return on saving and in fact:

$$(9) s(r_{t+1}) = 1/(2+\rho)$$

In addition, with Cobb-Douglas production – $y_t = k_t^\alpha$ – the real wage is $w_t = k_t^\alpha - k_t \alpha k_t^{\alpha-1} = (1-\alpha)k_t^\alpha$.

Thus equation (8) becomes:

$$(10) k_{t+1} = \frac{1}{(1+n)(1+g)} \cdot \left[\frac{1}{2+\rho} \cdot (1-\alpha)k_t^\alpha \right]$$

We need to compare this with equation (2) in Problem 2.14 – the analogous expression in the discrete-time Solow model – with the additional assumption of 100% depreciation (i.e. $\delta = 1$).

The saving rate in this economy is total saving divided by total output. Note that this is not the same as $s(r_{t+1})$, which is simply the fraction of their labor income that the young save. Denote the economy's total saving rate as $\hat{s}$. Then $\hat{s}$ will equal the saving of the young plus the dissaving of the old, all divided by total output and we can do this all in units of effective labor.

The saving of the young is $[1/(2 + \rho)] \cdot (1 - \alpha) \cdot k_t^\alpha$. Since there is 100% depreciation, the old do not get to dissave by the amount of the capital stock -- there is no dissaving by the old. Thus:

$$\hat{s} = \frac{[1/(2 + \rho)] \cdot (1 - \alpha) \cdot k_t^\alpha}{k_t^\alpha} = \frac{1}{2 + \rho} \cdot (1 - \alpha)$$

Thus equation (10) can be rewritten as:

$$(11) \quad k_{t+1} = \frac{1}{(1 + n)(1 + g)} \cdot \hat{s} \cdot k_t^\alpha = \left[\frac{\hat{s}}{(1 + n)(1 + g)} \right] f(k_t)$$

Note that this is exactly the same as the expression for k_{t+1} as a function of k_t in the discrete-time Solow model with $\delta = 1$. That is, it is equivalent to equation (2) in Problem 2.14 with δ set to 1. Thus that version of the Solow model does have some microeconomic foundations, although the assumption of 100% depreciation is quite unrealistic.

Problem 2.16

a) Pay-As-You-Go Social Security

i) The utility function is given by:

$$(1) \quad \ln C_{1,t} + [1/(1 + \rho)] \cdot \ln C_{2,t+1}$$

With the social security tax of T per person, the individual faces the following constraints (with g -- the growth rate of technology -- equal to 0, A is simply a constant throughout):

$$(2) \quad C_{1,t} + S_t = Aw_t - T$$

$$(3) \quad C_{2,t+1} = (1 + r_{t+1})S_t + (1 + n)T$$

where S_t represents the individual's saving in the first period. As far as the individual is concerned, the rate of return on social security is $(1 + n)$, which in general will not be equal to the return on private saving which is $(1 + r_{t+1})$. From equation (3), $(1 + r_{t+1})S_t = C_{2,t+1} - (1 + n)T$. Solving for S_t yields:

$$(4) \quad S_t = \frac{C_{2,t+1}}{1 + r_{t+1}} - \frac{(1 + n)}{(1 + r_{t+1})} T$$

Now substitute equation (4) into equation (2):

$$C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = Aw_t - T + \frac{(1 + n)}{(1 + r_{t+1})} T$$

Rearranging, we get the intertemporal budget constraint:

$$(5) \quad C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = Aw_t - \frac{(r_{t+1} - n)}{(1 + r_{t+1})} T$$

We know that with logarithmic utility, the individual will consume fraction $(1 + \rho)/(2 + \rho)$ of her lifetime wealth in the first period. Thus:

$$(6) \quad C_{1,t} = \left(\frac{1 + \rho}{2 + \rho} \right) \cdot \left[Aw_t - \frac{(r_{t+1} - n)}{(1 + r_{t+1})} T \right]$$

To solve for saving per person, substitute equation (6) into equation (2):

$$S_t = Aw_t - \left(\frac{1+\rho}{2+\rho} \right) \left[Aw_t - \left(\frac{r_{t+1}-n}{1+r_{t+1}} \right) T \right] - T \Rightarrow S_t = \left[1 - \left(\frac{1+\rho}{2+\rho} \right) \right] Aw_t - \left[1 - \left(\frac{1+\rho}{2+\rho} \right) \cdot \left(\frac{r_{t+1}-n}{1+r_{t+1}} \right) \right] T$$

$$(7) S_t = \left[1/(2+\rho) \right] Aw_t - \left[\frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} \right] T$$

Note that if $r_{t+1} = n$, saving is reduced one-for-one by the social security tax. If $r_{t+1} > n$, saving falls less than one-for-one. Finally, if $r_{t+1} < n$, saving falls more than one-for-one.

Denote $Z_t \equiv [(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)] / (2+\rho)(1+r_{t+1})$ and thus equation (7) becomes:

$$(8) S_t = \left[1/(2+\rho) \right] Aw_t - Z_t T$$

It is still true that the capital stock in period $t+1$ will be equal to the total saving of the young in period t , hence:

$$(9) K_{t+1} = S_t L_t$$

Converting this into units of effective labor by dividing both sides of (9) by AL_{t+1} and using equation (8) yields:

$$(10) k_{t+1} = [1/(1+n)] \cdot \left[\left(1/(2+\rho) \right) w_t - Z_t T/A \right]$$

With a Cobb-Douglas production function, the real wage is given by:

$$(11) w_t = (1-\alpha)k_t^\alpha$$

Substituting (11) into (10) gives the new relationship between capital in period $t+1$ and capital in period t , all in units of effective labor:

$$(12) k_{t+1} = [1/(1+n)] \cdot \left[\left(1/(2+\rho) \right) \cdot (1-\alpha)k_t^\alpha - Z_t T/A \right]$$

ii) To see what effect the introduction of the social security system has on the balanced-growth-path value of k , we must determine the sign of Z_t . If it is positive, the introduction of the tax, T , shifts down the k_{t+1} curve and reduces the balanced-growth-path value of k .

$$Z_t = \frac{(2+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})} = \frac{(1+1+\rho)(1+r_{t+1}) - (1+\rho)(r_{t+1}-n)}{(2+\rho)(1+r_{t+1})}$$

simplifying further allows us to sign Z_t :

$$Z_t = \frac{(1+r_{t+1}) + (1+\rho)[(1+r_{t+1}) - (r_{t+1}-n)]}{(2+\rho)(1+r_{t+1})} = \frac{(1+r_{t+1}) + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})} > 0$$

Thus, the k_{t+1} curve shifts down — relative to the case without the social security — and k^* is reduced.

iii) If the economy is initially dynamically efficient, a marginal increase in T results in a gain to the old generation who receive the extra benefits. However, it reduces k^* further below k_{GR} and thus leaves future generations worse off, with lower consumption possibilities. If the economy was initially dynamically inefficient — so that $k^* > k_{GR}$ — the old generation again gains due to the extra benefits. Now, the reduction in k^* actually allows for higher consumption for future generations and is welfare-improving. The introduction of the tax in this case reduces or may possibly eliminate the dynamic inefficiency caused by the over-accumulation of capital.

b) Fully-Funded Social Security

i) Equation (3) becomes:

$$(13) C_{2,t+1} = (1+r_{t+1})S_t + (1+r_{t+1})T$$

As far as the individual is concerned, the rate of return on social security is the same as that on private saving. We can now derive the intertemporal budget constraint. From equation (16):

$$(14) S_t = C_{2,t+1} / (1 + r_{t+1}) - T$$

Substituting equation (14) into equation (2) yields:

$$C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = Aw_t - T + T$$

or simply:

$$(15) C_{1,t} + \frac{C_{2,t+1}}{1 + r_{t+1}} = Aw_t$$

This is just the usual intertemporal budget constraint in the Diamond model. Solving the individual's maximization problem yields the usual Euler equation:

$$C_{2,t+1} = [1/(1 + \rho)] \cdot (1 + r_{t+1}) \cdot C_{1,t}$$

Substituting this into the budget constraint — equation (15) — yields:

$$(16) C_{1,t} = [(1 + \rho)/(2 + \rho)] Aw_t$$

To get saving per person, substitute equation (16) into equation (2):

$$S_t = Aw_t - [(1 + \rho)/(2 + \rho)] Aw_t - T$$

or simply:

$$(17) S_t = [1/(2 + \rho)] Aw_t - T$$

The social security tax causes a one-for-one reduction in private saving.

The capital stock in period $t+1$ will be equal to the sum of total private saving of the young plus the total amount invested by the government, hence:

$$(18) K_{t+1} = S_t L_t + TL_t$$

Dividing both sides of (18) by AL_{t+1} to convert this into units of effective labor, and using equation (17) yields:

$$k_{t+1} = \left(\frac{1}{1+n} \right) \cdot \left[\left(\frac{1}{2+\rho} \right) w_t - \frac{T}{A} \right] + \left(\frac{1}{1+n} \right) \cdot \frac{T}{A}$$

which simplifies to:

$$k_{t+1} = [1/(1+n)] \cdot [1/(2+\rho)] \cdot w_t$$

Using equation (11) to substitute for the wage yields:

$$(19) k_{t+1} = [1/(1+n)] \cdot [1/(2+\rho)] \cdot (1-\alpha)k_t^\alpha$$

Thus the fully-funded social security system has no effect on the relationship between the capital stock in successive periods.

ii) Since there is no effect on the relationship between k_{t+1} and k_t , the balanced-growth-path value of k is the same as it was before the introduction of the fully funded social security system. (Note that we have been assuming that the amount of the tax is not greater than the amount of saving each individual would have done in the absence of the tax). The basic idea is that total investment and saving is still the same each period — the government is simply doing some of the saving for the young. Since social security pays the same rate of return as private saving, individuals are indifferent as to who does the saving. Thus individuals offset one-for-one any saving that the government does for them.

Problem 2.17

a) In the decentralized equilibrium, there will be no intergenerational trade. Even if the young would like to trade goods this period for goods next period, the only people around to trade with are the old. Unfortunately, the old will be dead — and thus in no position to complete the trade — next period.

The individual's utility function is given by:

$$(1) \ln C_{1,t} + \ln C_{2,t+1}$$

The constraints are:

$$(2) C_{1,t} + F_t = A \quad \text{and} \quad (3) C_{2,t+1} = xF_t$$

where F_t is the amount stored by the individual.

Substituting equation (3) into (2) yields the intertemporal budget constraint:

$$(4) C_{1,t} + C_{2,t+1}/x = A$$

The individual's problem is to maximize lifetime utility – as given by equation (1) – subject to the intertemporal budget constraint – as given by equation (4). Set up the Lagrangian:

$$\mathcal{L} = \ln C_{1,t} + \ln C_{2,t+1} + \lambda [A - C_{1,t} - C_{2,t+1}/x]$$

The first-order conditions are given by:

$$\partial \mathcal{L} / \partial C_{1,t} = 1/C_{1,t} - \lambda = 0 \Rightarrow 1/C_{1,t} = \lambda \quad (5)$$

$$\partial \mathcal{L} / \partial C_{2,t+1} = 1/C_{2,t+1} - \lambda/x = 0 \Rightarrow 1/C_{2,t+1} = \lambda/x \quad (6)$$

Substitute equation (5) into equation (6) and rearrange to obtain:

$$(7) C_{2,t+1} = xC_{1,t}$$

Substitute equation (7) into the intertemporal budget constraint – equation (4) – to obtain:

$$C_{1,t} + xC_{1,t}/x = A$$

or simply:

$$(8) C_{1,t} = A/2$$

To obtain an expression for second-period consumption, substitute equation (8) into equation (7):

$$(9) C_{2,t+1} = xA/2$$

When young, each individual consumes half of her endowment and stores the other half – that is, $f_t = 1/2$. This allows her to consume $xA/2$ when old. Note that with log utility, the fraction of her endowment that the individual stores does not depend upon the return to storage.

b) What is consumption per unit of effective labor at time t ? First, calculate total consumption at time t :

$$C_t = C_{1,t}N_t + C_{2,t}N_{t-1}$$

where there are N_t young and N_{t-1} old people alive at time t . Each young person consumes the fraction of their endowment that they do not store – $(1-f)A$ – while each old person gets to consume the gross return on the fraction of their endowment that they stored – fxA . Thus:

$$C_t = (1-f) \cdot A \cdot N_t + f \cdot x \cdot A \cdot N_{t-1}$$

To convert this into units of time t effective labor, divide both sides by AN_t to get:

$$C_t/AN_t = (1-f) + f[x/(1+n)]$$

Thus consumption per unit of time t effective labor is a weighted average of 1 and something less than 1, since $x < (1+n)$. It will therefore be maximized when the weight on 1 is 1, that is, when $f = 0$. (You could also do this for consumption per person alive at time t which would not change the result here).

The decentralized equilibrium, with $f = 1/2$, is not Pareto efficient. Since intergenerational trade is not possible, people are "forced" into storage because that is the only way they can save and consume in old age. They must do this even if the return on storage, x , is low. However, at any point in time, a social planner could take 1 unit from each young person and give $(1+n)$ units to each old person since there are fewer of them. With $(1+n) > x$, this gives a better return than storage. Therefore, the social planner could raise welfare by taking the half of each generation's endowment that they were going to store and instead give it to the old. The planner could then do this each period. This allows people to still consume $A/2$ units

when young and now $(1+n)A/2$ units when old. This is greater than the $xA/2$ units of consumption when old that they would have had in the decentralized equilibrium with storage.

Problem 2.18

a) The individual has a utility function given by:

$$(1) \ln C_{1,t} + \ln C_{2,t+1}$$

and constraints —expressed in units of money — given by:

$$(2) P_t C_{1,t} = P_t A - P_t F_t - M_t^d$$

$$(3) P_{t+1} C_{2,t+1} = P_{t+1} x F_t + M_t^d$$

where M_t^d is nominal money demand and F_t is the amount stored.

One way of thinking about the problem is the following. The individual has two decisions to make. The first is to decide how much of her endowment to consume and how much to "save". Then she must decide how to save, through storage, by holding money or a combination of both. With log utility, we can separate the two decisions since the rate of return on "saving" will not affect the fraction of the first-period endowment that is saved. From Problem 2.17, we know she will consume half of the endowment in the first period, regardless of the rate of return on money or storage. Thus:

$$(4) C_{1,t} = A/2$$

What does she do with the other half? That will depend upon the gross rate of return on storage $-x-$ relative to the gross rate of return on money, which is P_t/P_{t+1} . It is P_t/P_{t+1} since the individual can sell 1 unit of consumption in period t and get P_t units of money. In period $t+1$, one unit of consumption costs P_{t+1} units of money and thus one unit of money will buy $1/P_{t+1}$ units of consumption. Thus the individual's P_t units of money will buy her P_t/P_{t+1} units of consumption in period $t+1$.

CASE 1 : $x > P_t/P_{t+1}$

She will consume half of her endowment, store the rest and not hold any money since the rate of return on money is less than the rate of return on storage. Thus:

$$C_{1,t} = A/2 \quad F_t = A/2 \quad M_t^d/P_t = 0 \quad C_{2,t+1} = xA/2$$

CASE 2 : $x < P_t/P_{t+1}$

Now storage is dominated by holding money. She will consume half of her endowment and then sell the rest for money.

$$C_{1,t} = A/2 \quad F_t = 0 \quad M_t^d/P_t = A/2 \quad C_{2,t+1} = [P_t/P_{t+1}] \cdot [A/2]$$

CASE 3 : $x = P_t/P_{t+1}$

Money and storage pay the same rate of return. She will consume half of her endowment and is then indifferent as to how much of the other half to store and how much of it to sell for money. Let $\alpha \in [0,1]$ be the fraction of saving that is in the form of money. Thus:

$$C_{1,t} = A/2 \quad F_t = (1-\alpha)A/2 \quad M_t^d/P_t = \alpha A/2 \quad C_{2,t+1} = xA/2 = [P_t/P_{t+1}] \cdot [A/2]$$

b) Equilibrium requires that aggregate real money demand equal aggregate real money supply.

We can derive expressions for both real money demand and supply in period t :

$$\text{aggregate real money demand} = N_t \cdot [A/2]$$

$$\text{aggregate real money supply} = [N_0/(1+n)]M/P_t = [N_t/(1+n)^{t+1}]M/P_t$$

The expression for aggregate real money supply uses the fact that in period 0, each old person -- and there are $[N_0/(1+n)]$ of them -- receives M units of money. The last step then uses the fact that since population grows at rate n , $N_t = (1+n)^t N_0$ and thus $N_0 = N_t/(1+n)^t$.

We can then use the equilibrium condition to solve for P_t :

$$N_t \cdot [A/2] = [N_t/(1+n)^{t+1}]M/P_t \Rightarrow P_t = 2M/[A(1+n)^{t+1}] \quad (5)$$

We can similarly derive expressions for real money demand and supply in period $t+1$:

$$\text{aggregate real money demand} = N_{t+1} \cdot [A/2] = (1+n) \cdot N_t \cdot [A/2]$$

$$\text{aggregate real money supply} = [N_t/(1+n)^{t+1}]M/P_{t+1}$$

We can then use the equilibrium condition to solve for P_{t+1} :

$$(1+n) \cdot N_t \cdot [A/2] = [N_t/(1+n)^{t+1}]M/P_{t+1} \Rightarrow P_{t+1} = 2M/[A(1+n)^{t+2}] \quad (6)$$

Dividing equation (6) by equation (5) yields:

$$P_{t+1}/P_t = 1/(1+n) \Rightarrow P_{t+1} = P_t/(1+n)$$

This analysis holds for all time periods $t \geq 0$ and so $P_{t+1} = P_t/(1+n)$ is an equilibrium. This shows that if money is introduced into a dynamically inefficient economy, storage can be eliminated. The monetary equilibrium will thus result in attainment of the "golden-rule" level of storage. See problem 2.17, part (b) for an explanation of why zero storage maximizes consumption per unit of effective labor.

c) This is the situation where $P_t/P_{t+1} = x$ -- the return on money is equal to the return on storage. In this case, individuals are indifferent as to how much of their saving to store and how much to hold in the form of money. Let $\alpha_t \in [0,1]$ be the fraction of saving held in the form of money in period t . We can again derive expressions for aggregate real money demand and supply in period t :

$$\text{aggregate real money demand} = N_t \cdot \alpha_t [A/2]$$

$$\text{aggregate real money supply} = [N_0/(1+n)]M/P_t = [N_t/(1+n)^{t+1}]M/P_t$$

We can then use the equilibrium condition to solve for P_t :

$$N_t \cdot \alpha_t [A/2] = [N_t/(1+n)^{t+1}]M/P_t \Rightarrow P_t = 2M/[\alpha_t A(1+n)^{t+1}] \quad (7)$$

We can similarly derive expressions for real money demand and supply in period $t+1$:

$$\text{aggregate real money demand} = N_{t+1} \cdot \alpha_{t+1} [A/2] = (1+n) \cdot N_t \cdot \alpha_{t+1} [A/2]$$

$$\text{aggregate real money supply} = [N_t/(1+n)^{t+1}]M/P_{t+1}$$

We can then use the equilibrium condition to solve for P_{t+1} :

$$(1+n) \cdot N_t \cdot \alpha_{t+1} [A/2] = [N_t/(1+n)^{t+1}]M/P_{t+1} \Rightarrow P_{t+1} = 2M/[\alpha_{t+1} A(1+n)^{t+2}] \quad (8)$$

Dividing equation (8) by (7) yields:

$$P_{t+1}/P_t = [\alpha_t/\alpha_{t+1}] \cdot [1/(1+n)]$$

For $P_{t+1}/P_t = 1/x$, we need:

$$[\alpha_t/\alpha_{t+1}] \cdot [1/(1+n)] = 1/x \Rightarrow [\alpha_{t+1}/\alpha_t] = [x/(1+n)] < 1$$

Thus for all $t \geq 0$, $P_{t+1} = P_t/x$ will be an equilibrium for any path of α 's that satisfies $\alpha_{t+1}/\alpha_t = x/(1+n)$.

d) $P_t = \infty$ -- money is worthless -- is also an equilibrium. This occurs if the young generation at time 0 does not believe that money will be valued in the next period and thus that the generation 1 people will not

accept money for goods. In that case, in period 0, the young simply consume half of their endowment and store the rest, while the old have some useless pieces of paper to go along with their endowment. This is an equilibrium with real money demand equal to zero and real money supply equal to zero as well. If no one believes the next generation will accept money for goods, this equilibrium continues for all future time periods.

This will be the only equilibrium if the economy ends at some date T . The young at date T will not want to sell any of their endowment. They will maximize the utility of their one-period life by consuming all of their endowment in period T . Thus, if the old at date T held any money, they would be stuck with it and it would be useless to them. Thus when they are young -- in period $T - 1$ -- they will not sell any of their endowment for money, knowing that the money will be of no use to them when old. Thus, if the old at date $T - 1$ held any money, they would be stuck with it and it would be useless to them. Thus the old at $T - 1$ will not want any money when they are young and so on. Working backwards, no one would ever want to sell goods for money and money would not be valued.

Problem 2.19

a) i) The individual has a utility function given by:

$$(1) U = \ln C_{1,t} + \ln C_{2,t+1}$$

and a lifetime budget constraint given by:

$$(2) Q_t C_{1,t} + Q_{t+1} C_{2,t+1} = Q_t (A - S_t) + Q_{t+1} x S_t$$

From Problem 2.17, we know that with log utility, the individual wants to consume $A/2$ in the first period. How the individual accomplishes this depends on the gross rate of return on storage, which is x , relative to the gross rate of return on trading.

The individual can sell 1 unit of the good in period t for Q_t . In period $t + 1$, it costs Q_{t+1} to obtain 1 unit of the good or equivalently, it costs 1 to obtain $1/Q_{t+1}$ units of the good. Thus for Q_t , it is possible to obtain Q_t/Q_{t+1} units of the good. Thus selling a unit of the good in period t allows the individual to buy Q_t/Q_{t+1} units of the good in period $t + 1$. Thus the gross rate of return on trading is Q_t/Q_{t+1} .

Now, $Q_{t+1} = Q_t/x$ for all $t > 0$ is equivalent to $x = Q_t/Q_{t+1}$ for all $t > 0$. In other words, the rate of return on storage is equal to the rate of return on trading and hence the individual is indifferent as to how much to store and how much to trade. Let $\alpha_t \in [0, 1]$ represent the fraction of "saving", $A/2$, that the individual sells in period t . That is, the individual sells $\alpha_t (A/2)$ in period t . This allows the individual to buy the amount $\alpha_t (Q_t/Q_{t+1}) (A/2)$ when they are old in period $t + 1$. The individual stores a fraction $(1 - \alpha_t)$ of her "saving". Thus:

$$(3) S_t = (1 - \alpha_t)(A/2)$$

Consumption in period $t + 1$ will be equal to the amount the individual buys plus the amount they have through storage. Thus:

$$(4) C_{2,t+1} = \alpha_t (Q_t/Q_{t+1})(A/2) + (1 - \alpha_t)x(A/2)$$

Since we are considering a case where $Q_t/Q_{t+1} = x$, equation (5) can be rewritten as:

$$(5) C_{2,t+1} = \alpha_t x(A/2) + (1 - \alpha_t)x(A/2) = x(A/2)$$

Consider some period $t + 1$ and let N represent the total number of individuals born each period, which is constant. Aggregate supply in period $t + 1$ is equal to the total number of young individuals, N , multiplied by the amount that each young individual wishes to sell, $\alpha_{t+1} (A/2)$. Thus:

$$(6) \text{Aggregate Supply}_{t+1} = N\alpha_{t+1} (A/2)$$

Aggregate demand in period $t + 1$ is equal to the total number of old individuals, N , multiplied by the amount each old individual wishes to buy, $(Q_t/Q_{t+1})\alpha_t (A/2)$. Thus:

$$(7) \text{Aggregate Demand}_{t+1} = N(Q_t/Q_{t+1})\alpha_t (A/2)$$

For the market to clear, aggregate supply must equal aggregate demand:

$$N\alpha_{t+1} (A/2) = N(Q_t / Q_{t+1}) \alpha_t (A/2)$$

or simply:

$$(8) \alpha_{t+1} = (Q_t / Q_{t+1}) \alpha_t$$

Since the proposed price path has $Q_{t+1} = Q_t / x$, the equilibrium condition given by equation (10) can also be written as:

$$(9) \alpha_{t+1} = x\alpha_t$$

Now consider the situation in period 0. The old individuals simply consume their endowment. Thus we must have α_0 equal to zero in order for the market to clear in period 0. Thus equation (10) implies that we must have $\alpha_t = 0$ for all $t \geq 0$.

The resulting equilibrium is the same as that in part (a) of Problem 2.17. The individual consumes half of her endowment in the first period of life, stores the rest and consumes $xA/2$ in the second period of life. Note that with $x < 1 + n$ here (since $n = 0$ and $x < 1$), this equilibrium is dynamically inefficient. Thus eliminating incomplete markets by allowing individuals to trade before the start of time does not eliminate dynamic inefficiency.

a) ii) Suppose the auctioneer announces $Q_{t+1} < Q_t / x$ or equivalently $x < (Q_t / Q_{t+1})$ for some date t . This means that trading dominates storage for the young at date t . This means that the young at date t will want to sell all of their saving — $\alpha_t = 1$ so that they want to sell $A/2$ — and not store anything. Thus aggregate supply in period t is equal to $N(A/2)$. For the old at date t , Q_{t+1} is irrelevant. They based their decision of how much to buy when old on Q_t / Q_{t-1} which was equal to x . Thus as described in part (a) (i), old individuals were not planning to buy anything. Thus aggregate demand in period t is zero. Thus aggregate demand will be less than aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

Suppose instead that the auctioneer announces $Q_{t+1} > Q_t / x$ or equivalently $x > (Q_t / Q_{t+1})$ for some date t . This means that storage dominates trading for the young at date t . This means that the young at date t will want to store their entire endowment and will want to buy $A/2$. For the old at date t , Q_{t+1} is irrelevant. They based their decision of how much to trade when old on Q_t / Q_{t-1} which was equal to x . Thus each old individual was not planning to buy or sell anything. Thus aggregate demand exceeds aggregate supply and the market for the good will not clear. Thus the proposed price path cannot be an equilibrium.

b) Consider the social planner's problem. The planner can divide the resources available for consumption between the young and the old in any manner. The planner can take, for example, one unit of each young person's endowment and transfer it to the old. Since there are the same number of old and young people in this model, this increases the consumption of each old person by 1. With $x < 1$, this method of transferring from the young to the old provides a better return than storage. If the economy did not end at some date T , the planner could prevent this change from making anyone worse off by requiring the next generation of young to make the same transfer in the following period. However, if the economy ends at some date T , the planner cannot do this. Taking anything from the young at date T would make them worse off since the planner cannot give them anything in return the next period — there is no next period. Thus the planner cannot make some generations better off without making another generation worse off. Thus the decentralized equilibrium is Pareto efficient.

c) It is infinite duration that is the source of the dynamic inefficiency. Allowing individuals to trade before the start of time requires a price path that results in an equilibrium which is equivalent to the situation where such a market does not exist. This equilibrium is not Pareto efficient — a social planner could raise

welfare by doing the procedure described in part (b). However, removing infinite duration also removes the social planner's ability to Pareto improve the decentralized equilibrium, as explained in part (b).

Problem 2.20

a) The individual has utility function given by:

$$(1) \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{C_{2,t+1}^{1-\theta}}{1-\theta} \quad \theta < 1$$

The constraints expressed in units of money are:

$$(2) P_t C_{1,t} = P_t A - M_t^d \quad \text{and} \quad (3) P_{t+1} C_{2,t+1} = M_t^d$$

Combining equations (2) and (3) yields the lifetime budget constraint:

$$(4) P_t C_{1,t} + P_{t+1} C_{2,t+1} = P_t A$$

Note that $\theta < 1$ means that the elasticity of substitution, $1/\theta$, is greater than one. Thus when the rate of return on saving increases, the substitution effect wins out — the individual will consume less now and save more. This is essentially what we will be showing here. As the rate of return on holding money — which is P_t/P_{t+1} — rises, the individual wishes to hold more money.

The individual's problem is to maximize (1) subject to (4). Set up the Lagrangian:

$$\mathcal{L} = \frac{C_{1,t}^{1-\theta}}{1-\theta} + \frac{C_{2,t+1}^{1-\theta}}{1-\theta} + \lambda \cdot [P_t A - P_t C_{1,t} - P_{t+1} C_{2,t+1}]$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_{1,t}} = C_{1,t}^{-\theta} - \lambda P_t = 0 \quad \Rightarrow \quad \lambda = \frac{C_{1,t}^{-\theta}}{P_t} \quad (5)$$

$$\frac{\partial \mathcal{L}}{\partial C_{2,t+1}} = C_{2,t+1}^{-\theta} - \lambda P_{t+1} = 0 \quad \Rightarrow \quad C_{2,t+1}^{-\theta} = \lambda P_{t+1} \quad (6)$$

Substitute (5) into (6) to obtain:

$$C_{2,t+1}^{-\theta} = C_{1,t}^{-\theta} P_{t+1}/P_t \Rightarrow (C_{2,t+1}/C_{1,t})^\theta = P_t/P_{t+1} \Rightarrow C_{2,t+1}/C_{1,t} = (P_t/P_{t+1})^{1/\theta}$$

or simply:

$$(7) C_{2,t+1} = (P_t/P_{t+1})^{1/\theta} \cdot C_{1,t}$$

This is the Euler equation, which can now be substituted into the budget constraint (4):

$$P_t C_{1,t} + P_{t+1} (P_t/P_{t+1})^{1/\theta} C_{1,t} = P_t A$$

Dividing by P_t yields:

$$C_{1,t} + (P_{t+1}/P_t) \cdot (P_t/P_{t+1})^{1/\theta} C_{1,t} = A$$

simplifying yields:

$$C_{1,t} + (P_t/P_{t+1})^{(1-\theta)/\theta} C_{1,t} = A \quad \Rightarrow \quad C_{1,t} \cdot [1 + (P_t/P_{t+1})^{(1-\theta)/\theta}] = A$$

Thus consumption when young is given by:

$$(8) C_{1,t} = \frac{A}{1 + (P_t/P_{t+1})^{(1-\theta)/\theta}}$$

To get the amount of her endowment that the individual sells for money (in real terms), we can use equation (2), expressed in real terms:

$$(2') M_t^d/P_t = A - C_{1,t}$$

Substitute equation (8) into (2') to obtain:

$$\frac{M_t^d}{P_t} = A - \frac{A}{1 + (P_t/P_{t+1})^{(1-\theta)/\theta}} \Rightarrow \frac{M_t^d}{P_t} = A \cdot \left[1 - \frac{1}{1 + (P_t/P_{t+1})^{(1-\theta)/\theta}} \right]$$

Simplifying by getting a common denominator yields:

$$(9) \quad \frac{M_t^d}{P_t} = A \cdot \frac{(P_t/P_{t+1})^{(1-\theta)/\theta}}{1 + (P_t/P_{t+1})^{(1-\theta)/\theta}}$$

Dividing the top and bottom of the right-hand side of equation (9) by $(P_t/P_{t+1})^{(1-\theta)/\theta}$ yields:

$$(10) \quad \frac{M_t^d}{P_t} = \frac{A}{(P_t/P_{t+1})^{(\theta-1)/\theta} + 1}$$

Thus the fraction of her endowment that the individual sells for money is:

$$(11) \quad h_t = \frac{1}{(P_t/P_{t+1})^{(\theta-1)/\theta} + 1}$$

It is straightforward to show that the fraction of her endowment that the agent sells for money is an increasing function of the rate of return on holding money:

$$\frac{\partial h_t}{\partial (P_t/P_{t+1})} = \frac{-[(\theta-1)/\theta] \cdot (P_t/P_{t+1})^{[(\theta-1)/\theta]-1}}{[(P_t/P_{t+1})^{(\theta-1)/\theta} + 1]^2} > 0 \text{ for } \theta < 1$$

We can also show that as the rate of return on money goes to zero, the amount of her endowment that the individual sells for money goes to zero. Use equation (9) to rewrite h_t as:

$$h_t = \frac{(P_t/P_{t+1})^{(1-\theta)/\theta}}{1 + (P_t/P_{t+1})^{(1-\theta)/\theta}}$$

and thus:

$$\lim_{(P_t/P_{t+1}) \rightarrow 0} h_t = 0/(1+0) = 0$$

b) The constraints expressed in real terms are:

$$(2') \quad C_{1,t} = A - M_t^d/P_t \quad \text{and} \quad (3') \quad C_{2,t+1} = M_t^d/P_{t+1}$$

Since there is no population growth, we can normalize the population to 1 without loss of generality.

From (3'), a generation born at time t plans to buy M_t^d/P_{t+1} units of the good when they are old. Thus, the generation born at time 0 plan to buy M_0^d/P_1 units when they are old (in period 1). Use equation (10) to find M_0^d , substituting $t = 0$:

$$M_0^d = \frac{P_0 A}{(P_0/P_1)^{(\theta-1)/\theta} + 1} \Rightarrow \frac{M_0^d}{P_1} = \frac{[P_0/P_1] \cdot A}{(P_0/P_1)^{(\theta-1)/\theta} + 1} \quad (12)$$

From equation (2'), a generation born at time t plans to sell M_t^d/P_t units of the good for money. Thus, the generation born at time 1 plan to sell M_1^d/P_1 units of the good. Substituting $t = 1$ into equation (10) gives:

$$(13) \frac{M_1^d}{P_1} = \frac{A}{(P_1/P_2)^{(\theta-1)/\theta} + 1}$$

In order for the amount of the consumption good that the generation 0 people wish to buy with their money – given by equation (12) – to be equal to the amount of the consumption good that the generation 1 people wish to sell for money – given by equation (13) – we need:

$$\frac{[P_0/P_1] \cdot A}{(P_0/P_1)^{(\theta-1)/\theta} + 1} = \frac{A}{(P_1/P_2)^{(\theta-1)/\theta} + 1} \Rightarrow \frac{(P_0/P_1)^{(\theta-1)/\theta} + 1}{(P_1/P_2)^{(\theta-1)/\theta} + 1} = \frac{P_0}{P_1}$$

Now with $P_0/P_1 < 1$, we need:

$$(P_0/P_1)^{(\theta-1)/\theta} + 1 < (P_1/P_2)^{(\theta-1)/\theta} + 1 \Rightarrow (P_0/P_1)^{(\theta-1)/\theta} < (P_1/P_2)^{(\theta-1)/\theta}$$

and since $(\theta - 1)/\theta$ is negative, this implies:

$$\frac{P_1}{P_2} < \frac{P_0}{P_1} < 1$$

c) Iterating this reasoning forward, the rate of return on money will have to be falling over time. That is:

$$1 > \frac{P_0}{P_1} > \frac{P_1}{P_2} > \frac{P_2}{P_3} > \dots \quad \text{and so} \quad \frac{P_t}{P_{t+1}} \rightarrow 0$$

As shown in part a, this means that the fraction of the endowment that is sold for money will also go to zero. The economy approaches the situation where everyone consumes their entire endowment in the first period. This is an equilibrium path in the sense that every time period, markets will clear. Each period, the real money demand by the young will be equal to real money supplied by the old and they will both go to zero as t gets large.

d) If $P_0/P_1 > 1$, we obtain the opposite result for the path of prices. That is, P_t/P_{t+1} will rise over time:

$$1 < \frac{P_0}{P_1} < \frac{P_1}{P_2} < \frac{P_2}{P_3} < \dots \quad \text{and so} \quad \frac{P_t}{P_{t+1}} \rightarrow \infty$$

From equation (10) we can see that this means that the fraction of the endowment sold for money will go to 1. In other words, the economy approaches the situation where no one consumes anything in the first period and sells their entire endowment for money. Thus, total real money demand will go to A , the endowment of the young – we have normalized the population to 1. But with this path of prices, the price level goes to zero, which means that real money supplied by the old goes to infinity. Thus, this cannot represent an equilibrium path for the economy because there will be a time period when real money supply will exceed real money demand and the market will not clear.

SOLUTIONS TO CHAPTER 3

Problem 3.1

The production functions for output and new knowledge are given by:

$$(1) Y(t) = A(t)(1 - a_L)L(t) \quad \text{and} \quad (2) \dot{A}(t) = Ba_L^\gamma L(t)^\gamma A(t)^\theta \quad \theta < 1$$

a) On a balanced growth path, $\dot{A}(t)/A(t) = g_A^* = \gamma n/(1 - \theta)$ (3)

Dividing both sides of equation (2) by $A(t)$ yields:

$$(4) \dot{A}(t)/A(t) = Ba_L^\gamma L(t)^\gamma A(t)^{\theta-1}$$

Equating (3) and (4) yields:

$$Ba_L^\gamma L(t)^\gamma A(t)^{\theta-1} = \gamma n/(1 - \theta) \Rightarrow A(t)^{\theta-1} = \gamma n/(1 - \theta) Ba_L^\gamma L(t)^\gamma$$

Simplifying and solving for $A(t)$ yields:

$$(5) A(t) = \left[(1 - \theta) Ba_L^\gamma L(t)^\gamma / \gamma n \right]^{1/(\theta-1)}$$

b) Substitute equation (5) into equation (1):

$$Y(t) = \left[(1 - \theta) Ba_L^\gamma L(t)^\gamma / \gamma n \right]^{1/(\theta-1)} (1 - a_L)L(t) = \left[(1 - \theta) B / \gamma n \right]^{1/(\theta-1)} a_L^{\gamma/(\theta-1)} (1 - a_L)L(t)^{[\gamma/(\theta-1)]+1}$$

We can maximize the log of output with respect to a_L :

$$\ln Y(t) = \left[1/(\theta-1) \right] \ln \left[(1 - \theta) B / \gamma n \right] + \left[\gamma/(\theta-1) \right] \ln a_L + \ln(1 - a_L) + \left[\left(\gamma/(\theta-1) \right) + 1 \right] \ln L(t)$$

The first-order condition is given by:

$$\frac{\partial \ln Y(t)}{\partial a_L} = \frac{\gamma}{(1 - \theta)} \cdot \frac{1}{a_L} - \frac{1}{1 - a_L} = 0$$

Some simple algebra yields an expression for a_L^* :

$$(6) a_L^* = \frac{\gamma}{(1 - \theta) + \gamma}$$

The higher is θ — the importance of knowledge in the production of new knowledge — and the higher is γ — the importance of labor in the production of new knowledge — the more of the labor force should be employed in the knowledge sector.

Problem 3.2

Substituting the production function — $Y_i(t) = K_i(t)^\theta$ — into the capital accumulation equation —

$\dot{K}_i(t) = s_i Y_i(t)$ — yields:

$$(1) \dot{K}_i(t) = s_i K_i(t)^\theta \quad \theta > 1$$

Dividing both sides of equation (1) by $K_i(t)$ gives an expression for the growth rate of the capital stock, $g_{K,i}$:

$$(2) g_{K,i}(t) \equiv \dot{K}_i(t)/K_i(t) = s_i K_i(t)^{\theta-1}$$

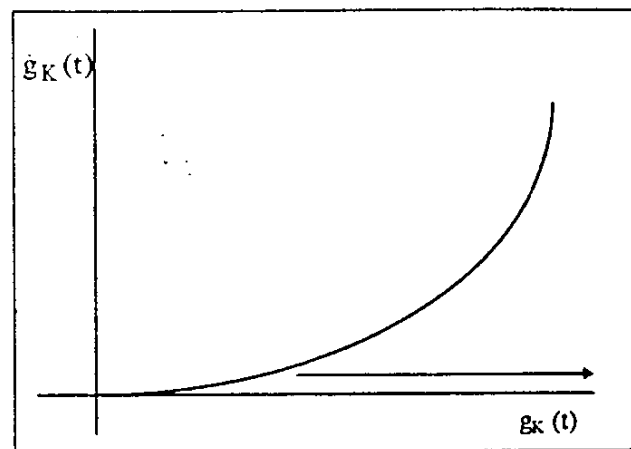
Taking the time derivative of the log of equation (2) yields an expression for the growth rate of the growth rate of capital:

$$(3) \dot{g}_{K,i}(t)/g_{K,i}(t) = (\theta - 1)g_{K,i}(t)$$

and thus:

$$(4) \dot{g}_{K,i}(t) = (\theta - 1)g_{K,i}(t)^2$$

Equation (4) is plotted at right. With $\theta > 1$, $g_{K,i}$ will be always increasing. The initial value of $g_{K,i}$ is determined by the initial capital stock and the saving rate — see equation (2). Since both economies have the same $K(0)$ but one has a higher saving rate, then from equation (2), the economy with the higher s will have the higher initial $g_{K,i}(0)$. From equation (3), the growth rate of $g_{K,i}$ is increasing in $g_{K,i}$. Thus the growth rate of the capital stock in the high saving economy will always exceed the growth rate of the capital stock in the low saving economy. That is, we



have $g_{K,1}(t) > g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two growth rates will be increasing over time. More formally, using the production function, we can write the ratio of output in the high-saving country — country 1 — to output in the low-saving country — country 2 — as:

$$(5) \quad Y_1(t)/Y_2(t) = [K_1(t)/K_2(t)]^\theta$$

Taking the time derivative of the log of equation (5) yields an expression for the growth rate of the ratio of output in the high-saving economy to output in the low-saving economy:

$$(6) \quad \frac{\dot{[Y_1(t)/Y_2(t)]}}{[Y_1(t)/Y_2(t)]} = \theta \left[\frac{\dot{K}_1(t)}{K_1(t)} - \frac{\dot{K}_2(t)}{K_2(t)} \right] = \theta [g_{K,1}(t) - g_{K,2}(t)] > 0$$

As explained above, $g_{K,1}(t)$ will exceed $g_{K,2}(t)$ for all $t \geq 0$. In fact, the gap between the two will be increasing over time. Thus the growth rate of the output ratio will be positive and increasing over time. That is, the ratio of output in the high-saving economy to output in the low-saving economy will be continually rising — and rising at an increasing rate.

Problem 3.3

The relevant equations are:

$$(1) \quad Y(t) = A(t)(1 - a_L)L \quad (2) \quad \dot{A}(t) = Ba_L LJ(t) \quad (3) \quad J(t) = \int_{\tau=0}^{\infty} (1 - e^{-v\tau}) \dot{A}(t - \tau) d\tau$$

a) Suppose that a possible solution for $A(t)$ is $A(t) = Ce^{gt}$. Taking the time derivative of this "guess" yields $\dot{A}(t) = Cge^{gt}$. Thus we can write the change in A at time $(t - \tau)$ as:

$$(4) \quad \dot{A}(t - \tau) = Cge^{g(t-\tau)}$$

Substituting equation (4) into equation (3) and solving the integral yields:

$$\begin{aligned} J(t) &= \int_{\tau=0}^{\infty} (1 - e^{-v\tau}) Cge^{g(t-\tau)} d\tau \\ J(t) &= \int_{\tau=0}^{\infty} Cge^{g(t-\tau)} d\tau - \int_{\tau=0}^{\infty} Cge^{g(t-(v+g)\tau)} d\tau = Cge^{gt} \int_{\tau=0}^{\infty} e^{-g\tau} d\tau - Cge^{gt} \int_{\tau=0}^{\infty} e^{-(v+g)\tau} d\tau \\ J(t) &= gA(t) \left[-\frac{1}{g} e^{-g\tau} \Big|_{\tau=0}^{\tau=\infty} \right] - gA(t) \left[-\frac{1}{v+g} e^{-(v+g)\tau} \Big|_{\tau=0}^{\tau=\infty} \right] = gA(t) \left[\frac{1}{g} \right] - gA(t) \left[\frac{1}{v+g} \right] \end{aligned}$$

which simplifies to:

$$(5) \quad J(t) = [v/(v+g)]A(t)$$

b) Is there actually a $g > 0$ such that $A(t) = Ce^{gt}$ — such that $\dot{A}(t)/A(t) = g$? Dividing both sides of equation (2) by $A(t)$ yields:

$$(6) \quad \dot{A}(t)/A(t) = Ba_L L J(t)/A(t)$$

Substituting equation (5) into equation (6) yields:

$$(7) \quad \dot{A}(t)/A(t) = Ba_L L v/(v + g)$$

For our guess to be correct, there must exist a $g > 0$ such that:

$$g = Ba_L L v/(v + g) \quad \Rightarrow \quad (v + g)g = Ba_L L v$$

or simply:

$$g^2 + vg - Ba_L L v = 0$$

Using the quadratic formula, we have:

$$g = \frac{-v \pm \sqrt{v^2 + 4Ba_L L v}}{2}$$

Clearly $v^2 + 4Ba_L L v > v^2$ and thus there will be a positive root given by:

$$(8) \quad g = \frac{-v + \sqrt{v^2 + 4Ba_L L v}}{2}$$

In order to see how g depends on v and what happens to g as v goes to infinity, we can rewrite equation (8) as:

$$g = \frac{(\sqrt{v^2 + 4Ba_L L v} - v) \cdot (\sqrt{v^2 + 4Ba_L L v} + v)}{2(\sqrt{v^2 + 4Ba_L L v} + v)} \Rightarrow g = \frac{v^2 + 4Ba_L L v - v^2}{2(\sqrt{v^2 + 4Ba_L L v} + 2v)}$$

Dividing the top and bottom by $2v$ yields:

$$(9) \quad g = \frac{2Ba_L L}{\sqrt{1 + 4Ba_L L/v} + 1}$$

Clearly, from equation (9), higher values of v imply higher values of g . That is, the sign of $\partial g/\partial v$ is positive. Taking the limit of the expression for g given in equation (9) as v goes to infinity yields:

$$\lim_{v \rightarrow \infty} g = \frac{2Ba_L L}{\sqrt{1 + 0} + 1} = Ba_L L$$

$J(t)$ represents the stock of knowledge that is useful in the production of new knowledge and it is not the entire stock of knowledge at time t . From equation (3), with $v > 0$, $J(t)$ is not simply the "sum" of all past increases in the stock of knowledge. When "summing up", changes in the stock of knowledge are discounted by $(1 - e^{-vt})$. Therefore, when calculating $J(t)$, recent additions to the stock of knowledge are discounted more heavily than additions to the stock of knowledge that occurred in the distant past. This is capturing the idea that it takes time for additions to the stock of knowledge to become useful in generating new knowledge. The larger is v , the less we discount recent additions to the stock of knowledge and the

more that $J(t)$ resembles $A(t)$. In fact, as $v \rightarrow \infty$, equation (3) simply becomes $J(t) = \int_{\tau=0}^{\infty} \dot{A}(t - \tau) d\tau$ and so

$J(t)$ is then just the "sum" of all past additions to the stock of knowledge and so $J(t) \rightarrow A(t)$ as $v \rightarrow \infty$.

Problem 3.4

The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ lines are given by:

$$(1) \quad \dot{g}_K = 0 \Rightarrow g_K = g_A + n$$

and:

$$(2) \quad \dot{g}_A = 0 \Rightarrow g_K = \frac{(1 - \theta)g_A - \gamma n}{\beta}$$

The expressions for the growth rates of capital and knowledge are:

$$(3) \quad g_K(t) = c_K [A(t)L(t)/K(t)]^{1-\alpha} \quad c_K \equiv s(1-a_K)^\alpha (1-a_L)^{1-\alpha}$$

$$(4) \quad g_A(t) = c_A K(t)^\beta L(t)^\gamma A(t)^{\theta-1} \quad c_A \equiv B a_K^\beta a_L^\gamma$$

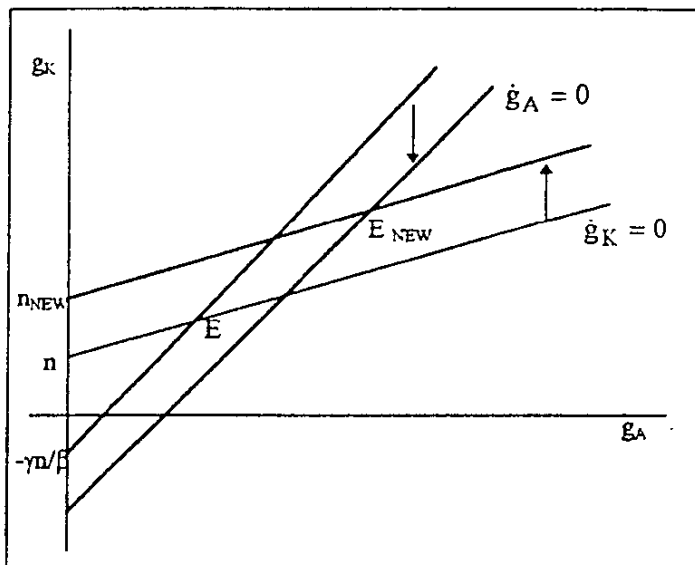
a) A rise in population growth from n to n_{NEW} :

From equation (1), for a given g_A , the value of g_K that satisfies $\dot{g}_K = 0$ is now higher.

Thus the $\dot{g}_K = 0$ locus shifts up. From equation (2), for a given g_A , the value of g_K that satisfies $\dot{g}_A = 0$ is now lower. Thus the $\dot{g}_A = 0$ locus shifts down.

Since n does not appear in equation (3), there is no jump in the value of g_K at the moment of the increase in population growth.

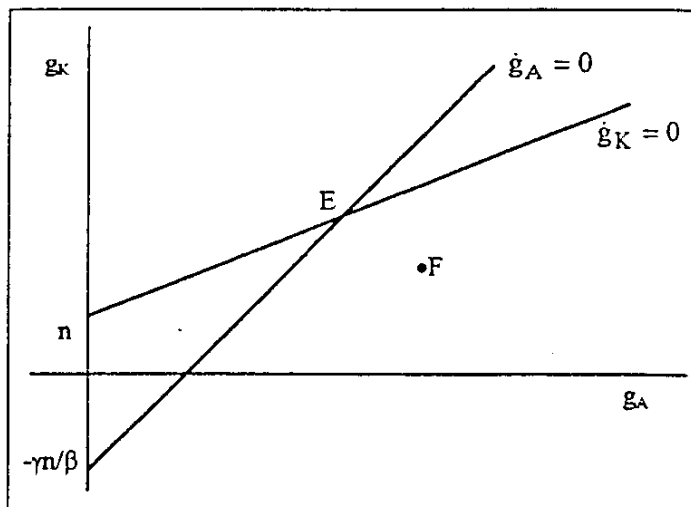
Similarly, since n does not appear in equation (4), there is no jump in the value of g_A at the moment of the rise in population growth.



b) An increase in the fraction of the capital stock used in the knowledge sector from a_K to a_K^{NEW} .

Note that a_K does not appear in equation (1) — the $\dot{g}_K = 0$ line — or in equation (2) — the $\dot{g}_A = 0$ line. Thus neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ line shifts as a result of the increase in a_K .

From equation (3), the rise in a_K causes the growth rate of capital, g_K , to jump down. From equation (4), the growth rate of knowledge, g_A , jumps up at the instant of the rise in a_K . Thus the economy moves to a point like F in the diagram.



c) A rise in θ , the coefficient on knowledge in the knowledge production function:

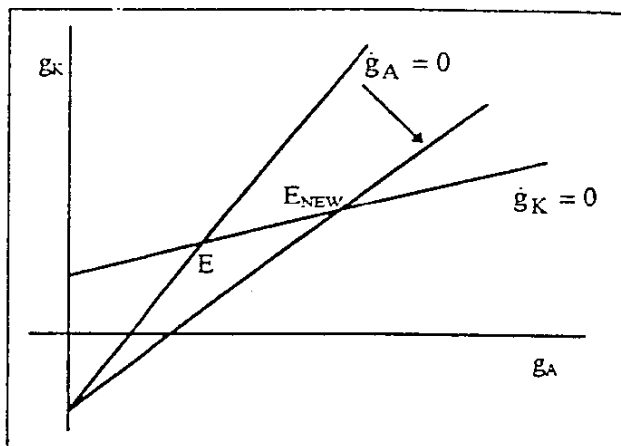
Since θ does not appear in equation (1), there is no shift of the $\dot{g}_K = 0$ locus. From equation (2), the $\dot{g}_A = 0$ locus has slope $(1 - \theta)/\beta$ and therefore becomes flatter after the rise in θ . See the diagram. Since θ does not appear in equation (3), the growth rate of capital, g_K , does not jump at the time of the rise in θ . θ does appear in equation (4) and thus we need to determine the effect that the rise in θ has on the growth rate of knowledge. It turns out that g_A may jump up, jump down or stay the same at the instant of the change in θ . Taking the log of both sides of equation (4) gives us:

$$\ln g_A(t) = \ln c_A + \beta \ln K(t) + \gamma \ln L(t) + (\theta - 1) \ln A(t)$$

Thus, taking the derivative with respect to θ yields:

$$(5) \quad \partial \ln g_A(t) / \partial \theta = \ln A(t)$$

So if $A(t)$ is less than one -- so that $\ln A(t) < 0$ -- the growth rate of knowledge jumps down at the instant of the rise in θ . However, if $A(t)$ is greater than one -- so that $\ln A(t) > 0$ -- the growth rate of knowledge jumps up at the instant of the rise in θ . Finally, if $A(t)$ is equal to one at the time of the change in θ , there is no initial jump in g_A . This means the dynamics of the adjustment to E_{NEW} may differ depending on the value of g_A at the time of the change in θ , but the end result is the same.



Problem 3.5

The equations of the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci are:

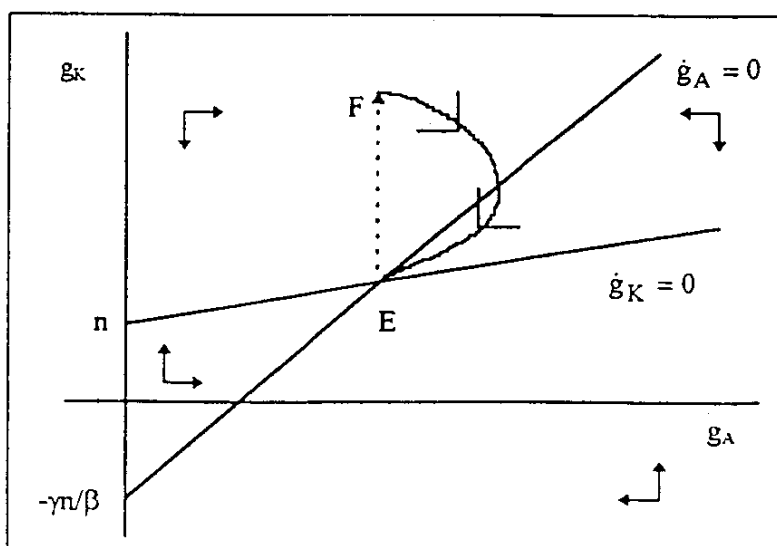
$$(1) \quad \dot{g}_K = 0 \Rightarrow g_K = g_A + n \quad (2) \quad \dot{g}_A = 0 \Rightarrow g_K = \frac{(1 - \theta)g_A - \gamma n}{\beta}$$

The equations defining the growth rates of capital and knowledge at any point in time are:

$$(3) \quad g_K(t) = c_K [A(t)L(t)/K(t)]^{1-\alpha} \quad c_K \equiv s(1 - a_K)^\alpha (1 - a_L)^{1-\alpha}$$

$$(4) \quad g_A(t) = c_A K(t)^\beta L(t)^\gamma A(t)^{\theta-1} \quad c_A \equiv B a_K^\beta a_L^\gamma$$

a) From equations (1) and (2) -- since the saving rate, s , does not appear -- neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ locus shifts when s increases. From equation (4), the growth rate of knowledge, g_A , does not change at the moment that s increases. However, from equation (3), a rise in s causes an upward jump in the growth rate of capital, g_K . In the diagram, the economy jumps from its balanced growth path at E to a point like F at the moment that s increases.



b) At point F , the economy is above the $\dot{g}_A = 0$ locus and thus g_A is rising.

Due to the increase in s , the growth rate of capital is higher than it would have been -- the amount of capital

going into the production of knowledge is higher than it would have been — the growth rate of knowledge begins to rise above what it would have been. Also at point F, the economy is above the $\dot{g}_K = 0$ and so g_K is falling. The economy drifts to the southeast and eventually crosses the $\dot{g}_A = 0$ locus at which point g_A begins to fall as well. Since there are decreasing returns to capital and knowledge in the production of new knowledge — $\theta + \beta < 1$ — the increase in s does not have a permanent effect on the growth rates of K and A . The economy eventually returns to point E.

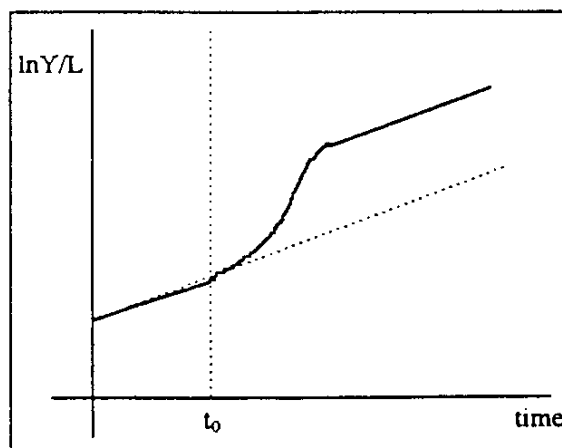
The production function is given by:

$$(5) \quad Y(t) = \left[(1 - a_K) K(t) \right]^\alpha \left[A(t) (1 - a_L) L(t) \right]^{1-\alpha}$$

Taking the time derivative of the log of equation (5) will yield the growth rate of total output:

$$(6) \quad \frac{\dot{Y}(t)}{Y(t)} = \alpha g_K(t) + (1 - \alpha) [g_A(t) + n]$$

On the initial balanced growth path, from equation (1), $g_K^* = g_A^* + n$. From equation (6), this means that total output is also growing at rate $g_K^* = g_A^* + n$ on the initial balanced growth path. Thus output per person, $Y(t)/L(t)$, is initially growing at rate g_A^* . During the transition period, both g_K and g_A are growing at a higher rate than on the balanced growth path — output per worker must also be growing at a rate greater than its balanced-growth-path value of g_A^* . Whether the



growth rate of output per worker is rising or falling will depend, among other things, on the value of α since there is a period of time when g_K is falling and g_A is rising. The diagram shows the growth rate of output per worker initially rising and then falling, but the important point is that during the entire transition, the growth rate itself is higher than its balanced-growth-path value of g_A^* . In the end, once the economy returns to point E, output per worker is again growing at rate g_A^* , which has not changed.

c) Note that the effects of an increase in s in this model are qualitatively similar to the effects in the Solow model. Since there are net decreasing returns to the produced factors of production here — $\theta + \beta < 1$ — the increase in s has only a level effect on output per worker. The path of output per worker lies above the path it would have taken but there is no permanent effect on the growth rate of output per worker, which on the balanced growth path is equal to the growth rate of knowledge. This is the same effect that a rise in s has in the Solow model where there are diminishing returns to the produced factor, capital. Quantitatively, the effect is larger than in the Solow model (for a given set of parameters). This is due to the fact that here, A rises above the path it would have taken while that is not true in the Solow model.

Problem 3.6

a) From equations (3.14) and (3.15) in the text, the growth rates of capital and knowledge are given by:

$$(1) \quad g_K(t) \equiv \dot{K}(t)/K(t) = c_K \left[A(t)L(t)/K(t) \right]^{1-\alpha} \quad \text{where } c_K \equiv s[1 - a_K]^\alpha [1 - a_L]^{1-\alpha}$$

$$(2) \quad g_A(t) \equiv \dot{A}(t)/A(t) = c_A K(t)^\beta L(t)^\gamma A(t)^{\theta-1} \quad \text{where } c_A \equiv B a_K^\beta a_L^\gamma$$

With the assumptions of $\beta + \theta = 1$ and $n = 0$, these equations simplify to:

$$(3) \quad g_K(t) = [c_K L^{1-\alpha}] \cdot [A(t)/K(t)]^{1-\alpha} \quad \text{and} \quad (4) \quad g_A(t) = [c_A L^\gamma] \cdot [K(t)/A(t)]^\beta$$

Thus given the parameters of the model and the population (which is the constant), the ratio A/K determines both growth rates. The two growth rates, g_K and g_A , will be equal when:

$$[c_K L^{1-\alpha}] \cdot [A(t)/K(t)]^{1-\alpha} = [c_A L^\gamma] \cdot [K(t)/A(t)]^\beta \Rightarrow [A(t)/K(t)]^{1-\alpha+\beta} = [c_A/c_K] \cdot L^{\gamma-(1-\alpha)}$$

Thus the value of A/K that yields equal growth rates of capital and knowledge is given by:

$$(5) \quad A(t)/K(t) = \left[(c_A/c_K) \cdot L^{\gamma-(1-\alpha)} \right]^{1/(1-\alpha+\beta)}$$

In order to find the growth rate of A and K when $g_K = g_A = g^*$, substitute equation (5) into equation (3):

$$g^* = \left[c_K L^{1-\alpha} \right] \cdot \left[(c_A/c_K) \cdot L^{\gamma-(1-\alpha)} \right]^{(1-\alpha)/(1-\alpha+\beta)}$$

Simplifying the exponents yields:

$$g^* = \left[c_K^{(1-\alpha+\beta)-(1-\alpha)} c_A^{1-\alpha} L^{(1-\alpha)+\gamma(1-\alpha)-(1-\alpha)^2} \right]^{1/(1-\alpha+\beta)}$$

or simply:

$$(6) \quad g^* = \left[c_K^\beta c_A^{1-\alpha} L^{(1-\alpha)(\gamma+\alpha)} \right]^{1/(1-\alpha+\beta)}$$

c) In order to see how an increase in s affects the long-run growth rate of the economy, substitute the definitions of c_K and c_A into equation (6):

$$(7) \quad g^* = \left[s^\beta (1-a_K)^{\alpha\beta} (1-a_L)^{(1-\alpha)\beta} B^{1-\alpha} a_K^{\beta(1-\alpha)} a_L^{\gamma(1-\alpha)} L^{(1-\alpha)(\gamma+\alpha)} \right]^{1/(1-\alpha+\beta)}$$

Taking the log of both sides of equation (7) gives us:

$$(8) \quad \ln g^* = \left[1/(1-\alpha+\beta) \right] \cdot \{ \beta \ln s + (1-\alpha) \cdot (\gamma+\alpha) \ln L + (1-\alpha) \ln B + \beta[\alpha \ln(1-a_K) + (1-\alpha) \ln a_K] + (1-\alpha)[\beta \ln(1-a_L) + \gamma \ln a_L] \}$$

Using equation (8), the elasticity of the long-run growth rate of the economy with respect to the saving rate is:

$$(9) \quad \partial \ln g^* / \partial \ln s = \beta / (1-\alpha+\beta) > 0$$

Thus an increase in the saving rate increases the long-run growth rate of the economy. This is essentially because it increases the resources devoted to physical capital accumulation and in this model, we have constant returns to the produced factors of production.

d) We can maximize $\ln g^*$ with respect to a_K to determine the fraction of the capital stock that should be employed in the R&D sector in order to maximize the long-run growth rate of the economy. The first-order condition is:

$$\frac{\partial \ln g^*}{\partial a_K} = \frac{\beta}{(1-\alpha+\beta)} \cdot \left[\frac{-\alpha}{(1-a_K)} + \frac{(1-\alpha)}{a_K} \right] = 0$$

Solving for the optimal a_K^* yields:

$$\alpha/(1-a_K) = (1-\alpha)/a_K \Rightarrow \alpha a_K = 1 - a_K + \alpha a_K - \alpha \Rightarrow 0 = 1 - a_K - \alpha$$

and thus:

$$(10) \quad a_K^* = (1-\alpha)$$

Thus the optimal fraction of the capital stock to employ in the R&D sector is equal to effective labor's share in the production of output. Note that β , capital's share in the production function for new knowledge, does not affect the optimal allocation of capital to the R&D sector. The reason for this is that an increase in β has two effects. It makes capital more important in the R&D sector, thereby tending to raise the a_K that maximizes g^* . A rise in β also makes the production of new capital more valuable, and new capital is produced when there is more output to be saved and invested. This tends to lower the a_K that maximizes g^* since it implies that more resources should be devoted to the production of output rather than knowledge. In the case we are considering, these two effects exactly cancel each other out.

Problem 3.7

The relevant equations are:

$$(1) Y(t) = K(t)^\alpha A(t)^{1-\alpha} \quad (2) \dot{K}(t) = sY(t) \quad (3) \dot{A}(t) = BY(t)$$

a) Substitute equation (1) into (2): $\dot{K}(t) = sK(t)^\alpha A(t)^{1-\alpha}$. Divide through by $K(t)$ to obtain an expression for the growth rate of capital, $g_K(t)$:

$$(4) g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1} A(t)^{1-\alpha}$$

Substitute equation (1) into (3): $\dot{A}(t) = BK(t)^\alpha A(t)^{1-\alpha}$. Divide through by $A(t)$ to obtain an expression for the growth rate of knowledge, $g_A(t)$:

$$(5) g_A(t) \equiv \dot{A}(t)/A(t) = BK(t)^\alpha A(t)^{-\alpha}$$

b) Capital

Taking the time derivative of equation (4) yields the growth rate of the growth rate of capital:

$$\frac{\dot{g}_K(t)}{g_K(t)} = (\alpha - 1) \frac{\dot{K}(t)}{K(t)} + (1 - \alpha) \frac{\dot{A}(t)}{A(t)}$$

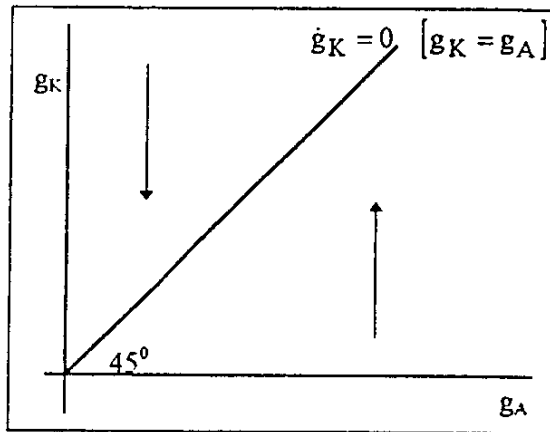
or:

$$(6) \dot{g}_K(t)/g_K(t) = (\alpha - 1)g_K(t) + (1 - \alpha)g_A(t)$$

From equation (6), g_K will be constant when

$(\alpha - 1)g_K + (1 - \alpha)g_A = 0$ or when $g_K = g_A$. Thus the $\dot{g}_K = 0$ locus is a 45° line in (g_A, g_K) space. Also, g_K will be rising when $(\alpha - 1)g_K + (1 - \alpha)g_A > 0$ or when $g_K < g_A$. Thus g_K is rising below the $\dot{g}_K = 0$ line.

Lastly, g_K will be falling when $(\alpha - 1)g_K + (1 - \alpha)g_A < 0$ or when $g_K > g_A$. Thus g_K is falling above the $\dot{g}_K = 0$ line.

**Knowledge**

Taking the time derivative of the log of equation (5) yields the growth rate of the growth rate of knowledge:

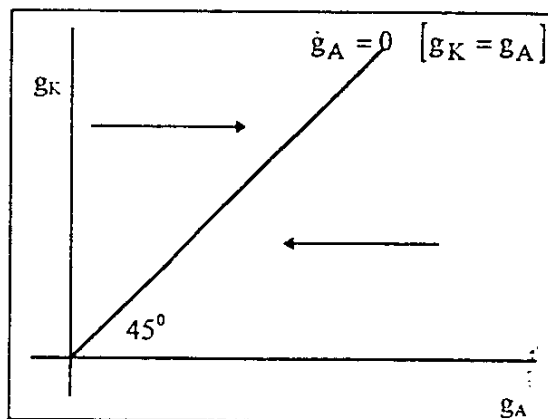
$$\frac{\dot{g}_A(t)}{g_A(t)} = \alpha \frac{\dot{K}(t)}{K(t)} - \alpha \frac{\dot{A}(t)}{A(t)}$$

or:

$$(7) \dot{g}_A(t)/g_A(t) = \alpha g_K(t) - \alpha g_A(t)$$

From equation (7), g_A will be constant when

$\alpha g_K - \alpha g_A = 0$ or when $g_K = g_A$. Thus the $\dot{g}_A = 0$ locus is also a 45° line in (g_A, g_K) space. Also, g_A will be rising when $\alpha g_K - \alpha g_A > 0$ or when $g_K > g_A$. Thus above the $\dot{g}_A = 0$ line, g_A will be rising. Finally, g_A will be falling when $\alpha g_K - \alpha g_A < 0$ or when $g_K < g_A$. Thus below the $\dot{g}_A = 0$ line, g_A will be falling.



c) Put the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci onto one diagram.

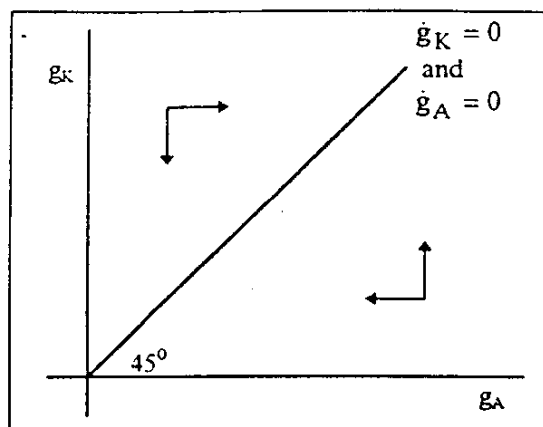
Although we can see that the economy will eventually arrive at a situation where $g_K = g_A$ and they are constant, we still do not have enough information to determine the unique balanced growth path.

Rewriting equations (4) and (5):

$$(4) \quad g_K(t) = sK(t)^{\alpha-1} A(t)^{1-\alpha} = s[A(t)/K(t)]^{1-\alpha}$$

$$(5) \quad g_A(t) = BK(t)^\alpha A(t)^{-\alpha} = B[A(t)/K(t)]^{-\alpha}$$

At any point in time, the growth rates of capital and knowledge are linked because they both depend on the ratio of knowledge to capital at that point in time. It is therefore possible to write one growth rate as a function of the other.



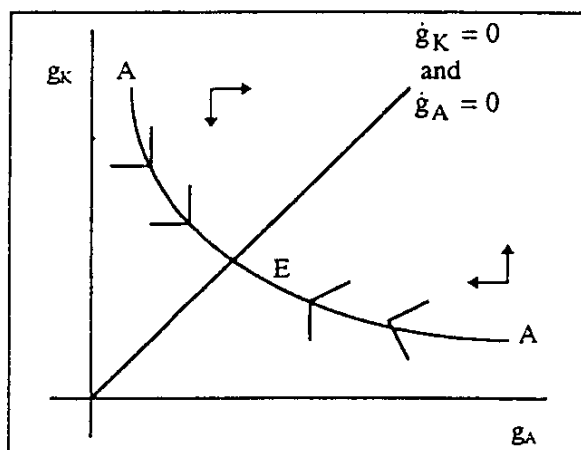
From equation (5), $[A(t)/K(t)]^\alpha = B/g_A(t)$ or simply:

$$(8) \quad A(t)/K(t) = [B/g_A(t)]^{1/\alpha}$$

Substitute equation (8) into equation (4):

$$(9) \quad g_K(t) = s[B/g_A(t)]^{(1-\alpha)/\alpha}$$

It must be the case that g_K and g_A lie on the locus satisfying equation (9) – labeled AA in the diagram. Regardless of the initial ratio of A/K the economy starts somewhere on this locus and then moves along it to point E. Thus the economy does converge to a unique balanced growth path at E.



To calculate the growth rates of capital and knowledge on the balanced growth path, note that at point E we are on the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci

where $g_K = g_A$. Letting g^* denote this common growth rate, then from equation (9), $g^* = s[B/g^*]^{(1-\alpha)/\alpha}$

Re-arranging to solve for g^* yields:

$$(10) \quad g^* = s^\alpha B^{1-\alpha}$$

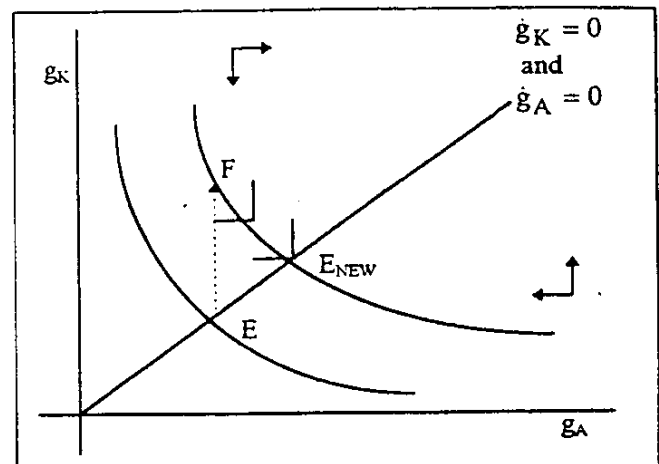
Taking the time derivative of the log of the production function, equation (1), yields the growth rate of real output, $\dot{Y}(t)/Y(t) = \alpha g_K(t) + (1-\alpha)g_A(t)$. On the balanced growth path, $g_K = g_A \equiv g^*$, and thus:

$$(11) \quad \dot{Y}(t)/Y(t) = \alpha g^* + (1-\alpha)g^* = g^* = s^\alpha B^{1-\alpha}$$

On the balanced growth path, capital, knowledge and output all grow at rate g^* .

d) Clearly, from equation (10), a rise in the saving rate, s , raises g^* and thus raises the long run growth rates of capital, knowledge and output.

From equations (6) and (7), neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ lines shift when s changes — s does not appear in either equation. From equation (4), a rise in s causes g_K to jump up. Also, the locus given by equation (9) shifts out. So at the moment that s rises, the economy moves from its balanced growth path at point E to a point like F . It then moves down along the equation (9) locus until it reaches a new balanced growth path at point E_{NEW} .



Problem 3.8

a) Taking the partial derivative of output at firm i — $Y_i = K_i^\alpha L_i^{1-\alpha} [K^\phi L^{-\phi}]$ — with respect to K_i , treating the aggregate capital stock as given yields:

$$(1) \quad r = \frac{\partial Y_i}{\partial K_i} = \alpha K_i^{\alpha-1} L_i^{1-\alpha} [K^\phi L^{-\phi}] = \alpha \left(\frac{K_i}{L_i} \right)^{\alpha-1} \left(\frac{K}{L} \right)^\phi$$

In equilibrium, the capital-labor ratio is equated across firms. Thus K_i/L_i must equal the economy-wide capital-labor ratio, which is K/L . Substituting this fact into equation (1) gives us the private marginal product of capital:

$$(2) \quad r = \alpha \left(\frac{K}{L} \right)^{\alpha-1+\phi}$$

b) We can employ the technique used to solve for the balanced growth path in the Solow model. Since K_i/L_i is the same across firms and the production function has constant returns, the aggregate production function is given by $Y = K^\alpha L^{1-\alpha} [K^\phi L^{-\phi}]$ or simply:

$$(3) \quad Y = K^{(\alpha+\phi)} L^{1-\alpha-\phi}$$

Define $k \equiv K/L$ and $y \equiv Y/L$. Dividing both sides of equation (3) by L gives us:

$$\frac{Y}{L} = \left(\frac{K}{L} \right)^{\alpha+\phi} \left(\frac{L}{L} \right)^{1-\alpha-\phi}$$

and thus output per worker is given by:

$$(4) \quad y = k^{\alpha+\phi}$$

Taking the time derivative of both sides of the definition of $k \equiv K/L$ yields:

$$(5) \quad \dot{k} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \left(\frac{K}{L} \right) \frac{\dot{L}}{L}$$

Substituting the capital accumulation equation — $\dot{K} = sY$ — and the assumption that the labor force grows at rate n — $\dot{L}/L = n$ — into equation (5) gives us:

$$(6) \quad \dot{k} = sY/L - nk = sy - nk$$

Substituting equation (4) for output per worker into equation (6) gives us:

$$(7) \quad \dot{k} = sk^{\alpha+\phi} - nk$$

Just as in the Solow model, the economy will converge to a situation where actual investment per worker $- sk^{\alpha+\phi}$ -- is equal to break-even investment per worker $- nk$. Thus on a balanced growth path, capital per worker will be constant. Setting $\dot{k} = 0$ gives us:

$$sk^{\alpha+\phi} = nk \quad \Rightarrow \quad k^{1-\alpha-\phi} = s/n$$

or simply:

$$(8) \quad k^* = [s/n]^{1/(1-\alpha-\phi)}$$

Substituting equation (8) into equation (2) yields:

$$r = \alpha[s/n]^{-(1-\alpha-\phi)/(1-\alpha-\phi)} = \alpha[s/n]^{-1}$$

and thus the marginal product of capital on the balanced growth path is:

$$(9) \quad r^* = \alpha n/s$$

c) The analysis above does not support the claim. The value of ϕ does not affect the steady-state value of the private marginal product of capital, r^* . In addition, ϕ does not affect how this value of r^* changes when the saving rate changes. From equation (9), $\partial r^*/\partial s = -(\alpha n)/s^2$, which does not depend on ϕ . That is, positive externalities from capital do not mitigate the decline in the marginal product of capital caused by a rise in the saving rate. Why? It is true, as the claim asserts, that a higher ϕ means that r responds less to changes in the capital-labor ratio, K/L -- see equation (2). However, it is also true that a higher ϕ means that the capital-labor ratio itself responds more to the change in the saving rate -- see equation (8). In this case, the effects cancel each other out.

Problem 3.9

The production functions -- after the normalization of $R = 1$ -- are given by:

$$(1) \quad C(t) = K_C(t)^\alpha \quad \text{and} \quad (2) \quad \dot{K}(t) = BK_K(t)$$

a) The return to employing an additional unit of capital in the capital producing sector is given by $\partial \dot{K}(t)/\partial K_K(t) = B$. This has value $P_K(t) \cdot B$ in units of consumption goods. The return from employing an additional unit of capital in the consumption producing sector is $\partial C(t)/\partial [K_C(t)] = \alpha [K_C(t)]^{\alpha-1}$.

Equating these returns gives us:

$$(3) \quad P_K(t) \cdot B = \alpha [K_C(t)]^{\alpha-1}$$

Taking the time derivative of the log of equation (3) yields the growth rate of the price of capital goods relative to consumption goods:

$$\frac{\dot{P}_K(t)}{P_K(t)} + \frac{\dot{B}}{B} = \frac{\dot{\alpha}}{\alpha} + (\alpha - 1) \left[\frac{\dot{K}_C(t)}{K_C(t)} \right] \Rightarrow \frac{\dot{P}_K(t)}{P_K(t)} = (\alpha - 1) \frac{\dot{K}_C(t)}{K_C(t)}$$

The last step uses the fact that B and α are constants. Now since $K_C(t)$ is growing at rate $g_K(t)$ and denoting the growth rate of $P_K(t)$ as $g_P(t)$, we have:

$$(4) \quad g_P(t) = (\alpha - 1)g_K(t)$$

b) i) The growth rate of consumption is given by:

$$(5) \quad g_C(t) \equiv \dot{C}(t)/C(t) = [r(t) - \rho]/\sigma = [B + g_P(t) - \rho]/\sigma = [B + (\alpha - 1)g_K(t) - \rho]/\sigma$$

where the last step uses equation (4) to substitute for $g_P(t)$.

b) ii) Taking the time derivative of the log of the consumption production function, equation(1), yields:

$$(6) \quad g_C(t) \equiv \dot{C}(t)/C(t) = \alpha [\dot{K}_C(t)/K_C(t)] = \alpha g_K(t)$$

Equating the two expressions for the growth rate of consumption -- equations (5) and (6) -- yields:

$$\alpha g_K(t) = [B + (\alpha - 1)g_K(t) - \rho]/\sigma \Rightarrow \alpha \sigma g_K(t) + (1 - \alpha)g_K(t) = B - \rho$$

Thus in order for C to be growing at rate $g_C(t)$, $K_C(t)$ must be growing at the following rate:

$$(7) \quad g_K(t) = (B - \rho)/[\alpha \sigma + (1 - \alpha)]$$

b) iii) We have already solved for $g_K(t)$ in terms of the underlying parameters. To solve for $g_C(t)$, substitute equation (7) into equation (6):

$$(8) \quad g_C(t) = \alpha(B - \rho)/[\alpha \sigma + (1 - \alpha)]$$

c) The real interest rate is now $(1 - \tau)(B + g_P)$. Thus equation (5) becomes:

$$(9) \quad g_C(t) = \frac{(1 - \tau) \cdot [B + g_P(t)] - \rho}{\sigma} = \frac{(1 - \tau) \cdot [B + (\alpha - 1)g_K(t)] - \rho}{\sigma}$$

where the last step uses equation (4) -- which is unaffected by the imposition of the tax -- to substitute for $g_P(t)$. Equating the two expressions for the growth rate of consumption -- equations (9) and (6) -- yields:

$$\alpha g_K(t) = \frac{(1 - \tau) \cdot [B + (\alpha - 1)g_K(t)] - \rho}{\sigma} \Rightarrow \alpha \sigma g_K(t) + (1 - \tau)(1 - \alpha)g_K(t) = (1 - \tau)B - \rho$$

and thus:

$$(10) \quad g_K(t) = \frac{(1 - \tau)B - \rho}{[\alpha \sigma + (1 - \tau)(1 - \alpha)]}$$

Substituting equation (10) into equation (6) yields an expression for the growth rate of consumption as a function of the underlying parameters of the model:

$$(11) \quad g_C(t) = \alpha \left[\frac{(1 - \tau)B - \rho}{\alpha \sigma + (1 - \tau)(1 - \alpha)} \right]$$

In order to see the effects of the tax, take the derivative of $g_C(t)$ with respect to τ :

$$\frac{\partial g_C(t)}{\partial \tau} = -\alpha \left\{ \frac{B[\alpha \sigma + (1 - \tau)(1 - \alpha)] - [(1 - \tau)B - \rho](1 - \alpha)}{[\alpha \sigma + (1 - \tau)(1 - \alpha)]^2} \right\} = -\alpha \left\{ \frac{B\alpha \sigma + \rho(1 - \alpha)}{[\alpha \sigma + (1 - \tau)(1 - \alpha)]^2} \right\} < 0$$

Thus an increase in the tax rate τ causes the growth rate of consumption to fall.

Problem 3.10

a) Note that the model of the northern economy is simply the Solow model with a constant growth rate of technology equal to $g = Ba_{LN}L_N$. From our analysis of the Solow model in Chapter 1, we know that the long-run growth rate of northern output per worker will be equal to that constant growth rate of technology.

b) Taking the time derivative of both sides of the definition, $Z(t) \equiv A_S(t)/A_N(t)$, yields:

$$(1) \quad \dot{Z}(t) = \frac{A_N(t)\dot{A}_S(t) - A_S(t)\dot{A}_N(t)}{A_N(t)^2}$$

Substituting the expressions for $\dot{A}_S(t)$ and $\dot{A}_N(t)$ into equation (1) gives us:

$$\dot{Z}(t) = \frac{A_N(t)[\mu a_{LS}L_S(A_N(t) - A_S(t))] - A_S(t)[Ba_{LN}L_N A_N(t)]}{A_N(t)^2}$$

Simplifying yields:

$$(2) \dot{Z}(t) = [\mu a_{LS} L_S (1 - A_S(t)/A_N(t))] - [A_S(t)/A_N(t)] \cdot [Ba_{LN} L_N]$$

Substituting the definition of $Z(t) \equiv A_S(t)/A_N(t)$ into equation (2) gives us:

$$\dot{Z}(t) = \mu a_{LS} L_S - \mu a_{LS} L_S Z(t) - Ba_{LN} L_N Z(t)$$

Collecting terms yields:

$$(3) \dot{Z}(t) = \mu a_{LS} L_S - [\mu a_{LS} L_S + Ba_{LN} L_N] Z(t)$$

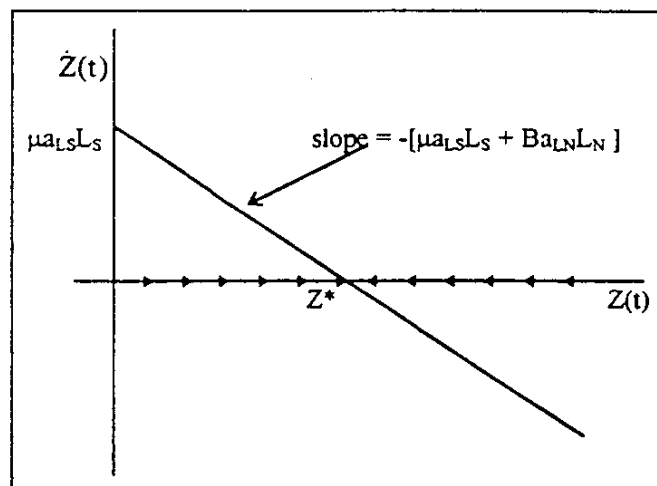
The phase diagram implied by equation (3) is depicted at right. Note that equation (3) and the accompanying phase diagram do not apply for the case of $Z \geq 1$, since $\dot{A}_S(t) = 0$ for $A_S(t) \geq A_N(t)$.

The relationship between $\dot{Z}(t)$ and $Z(t)$ is linear with slope equal to $-(\mu a_{LS} L_S + Ba_{LN} L_N) < 0$. From the phase diagram, if $Z < Z^*$, then $\dot{Z}(t) > 0$. Thus if Z begins to the left of Z^* , it rises toward Z^* over time. Similarly if $Z > Z^*$, then $\dot{Z}(t) < 0$. Thus if Z begins to the right of Z^* , it falls toward Z^* over time. Thus Z — the ratio of technology in the south to technology in the north — does converge to a stable value. To solve for Z^* , set $\dot{Z}(t) = 0$:

$$0 = \mu a_{LS} L_S - [\mu a_{LS} L_S + Ba_{LN} L_N] Z^*$$

Solving for Z^* yields:

$$(4) Z^* = \frac{\mu a_{LS} L_S}{\mu a_{LS} L_S + Ba_{LN} L_N}$$



The next step is to determine the long-run growth rate of southern output per worker. We have just shown that $Z(t) \equiv A_S(t)/A_N(t)$ converges to a constant. Thus in the long-run, $A_S(t)$ must be growing at the same rate as $A_N(t)$. In the long-run, then, the south is a Solow economy with a growth rate of technology equal to $Ba_{LN} L_N$. Thus the long-run growth rate of southern output per worker is equal to that growth rate.

Note that in the long-run, the growth rate of southern output per worker is the same as that in the north. This means that a_{LS} — the fraction of the south's labor force that is engaged in learning the technology of the north — does not affect the south's long-run growth rate. That growth rate is entirely determined by how many people the north has working to produce new technology.

c) Dividing the northern production function — $Y_N(t) = K_N(t)^\alpha [A_N(t)(1 - a_{LN})L_N]^{1-\alpha}$ — by the quantity of effective labor, $A_N(t)L_N$, yields:

$$(5) \frac{Y_N(t)}{A_N(t)L_N} = \left[\frac{K_N(t)}{A_N(t)L_N} \right]^\alpha \left[\frac{A_N(t)(1 - a_{LN})L_N}{A_N(t)L_N} \right]^{1-\alpha}$$

Defining output and capital per unit of effective labor as $y_N(t) \equiv Y_N(t)/A_N(t)L_N$ and $k_N(t) \equiv K_N(t)/A_N(t)L_N$ respectively, we can rewrite equation (5) as:

$$(6) y_N(t) = k_N(t)^\alpha (1 - a_{LN})^{1-\alpha}$$

Now we can use the technique employed to solve the Solow model to show that on the balanced growth path, $k_S^* = k_N^*$. Taking the time derivative of both sides of the definition of $k_N(t) \equiv K_N(t)/A_N(t)L_N$ yields:

$$(7) \dot{k}_N(t) = \frac{\dot{K}_N(t)}{A_N(t)L_N} - \frac{K_N(t)}{A_N(t)L_N} \cdot \frac{\dot{A}_N(t)}{A_N(t)}$$

Substituting the capital accumulation equation -- $\dot{K}_N(t) = s_N Y_N(t)$ -- into equation (7) gives us:

$$(8) \dot{k}_N(t) = \frac{s_N Y_N(t)}{A_N(t)L_N} - \frac{\dot{A}_N(t)}{A_N(t)} \cdot \frac{K_N(t)}{A_N(t)L_N} = s_N y_N(t) - Ba_{LN} L_N k_N(t)$$

where the last step uses the definitions of $y_N(t)$ and $k_N(t)$ and substitutes for the growth rate of northern technology. Finally, using equation (8) to substitute for $y_N(t)$ yields:

$$(9) \dot{k}_N(t) = s_N k_N(t)^\alpha (1 - a_{LN})^{1-\alpha} - Ba_{LN} L_N k_N(t)$$

An analogous derivation for the south will yield:

$$(10) \dot{k}_S(t) = s_S k_S(t)^\alpha (1 - a_{LS})^{1-\alpha} - Ba_{LN} L_N k_S(t)$$

where we have used the fact that in the long-run, the growth rate of technology in the south is $Ba_{LN} L_N$.

Using the facts that $s_N = s_S$ and $a_{LN} = a_{LS}$, we can see that the equations for the dynamics of k are the same for the two economies. Thus we know that the balanced-growth-path values of k and y will be the same for the two economies. That is, we know that $k_S^* = k_N^*$ and $y_S^* = y_N^*$. This implies:

$$(11) y_S^*/y_N^* = 1$$

Using the definitions of y_S and y_N , this implies that:

$$\frac{Y_S/A_S L_S}{Y_N/A_N L_N} = 1 \Rightarrow \frac{Y_S/L_S}{Y_N/L_N} = \frac{A_S}{A_N} \quad (12)$$

Equation (12) states that, on the balanced growth path, the ratio of output per worker in the south to output per worker in the north is equal to the ratio of technology in the south to technology in the north. From part (b), we know that A_S/A_N converges to Z^* in the long-run. Using equation (4) for Z^* to substitute into equation (12) leaves us with:

$$(13) \frac{Y_S/L_S}{Y_N/L_N} = \frac{\mu a_{LS} L_S}{\mu a_{LS} L_S + Ba_{LN} L_N}$$

Note that with $Ba_{LN} L_N > 0$, this ratio must be less than 1 -- output per person in the south will be lower than output per person in the north. Also note that on the balanced growth path, the ratio of output per person in the south to that in the north does depend on a_{LS} -- the fraction of southern workers engaged in learning the north's technology. In fact, the higher is a_{LS} , the closer will be the path of output per person in the south to that in the north.

Problem 3.11

a) We need to find a value of τ such that $[Y_N(t)/L_N]/[Y_S(t)/L_S]$ -- the ratio of output per worker in the north to that in the south -- is equal to 10. From the northern production function:

$$(1) Y_N(t)/L_N = A_N(t)(1 - a_L)$$

Taking the time derivative of the natural log of equation (1) yields an expression for the growth rate of northern output per worker:

$$(2) \frac{[Y_N(t)/L_N] \cdot \dot{}}{Y_N(t)/L_N} = \frac{\dot{A}_N(t)}{A_N(t)} = 0.03$$

The last step uses the information given in the problem that the growth rate of northern output per worker -- and thus of northern knowledge -- is 3% per year. Since $\dot{A}_N(t)/A_N(t) = 0.03$ then:

$$(3) A_N(t) = e^{0.03\tau} A_N(t - \tau)$$

From the southern production function:

$$(4) Y_S(t)/L_S = A_S(t)$$

Dividing equation (3) by equation (4) yields an expression for the ratio of output per worker in the north to that in the south:

$$(5) \frac{Y_N(t)/L_N}{Y_S(t)/L_S} = \frac{A_N(t)(1-a_L)}{A_S(t)} \approx \frac{A_N(t)}{A_N(t-\tau)} = e^{0.03\tau}$$

The second to last step uses the fact that $a_L \approx 0$ and that $A_S(t) = A_N(t-\tau)$. The last step uses equation (3).

Thus we need a τ such that:

$$e^{0.03\tau} = 10 \Rightarrow 0.03\tau = \ln(10) \Rightarrow \tau = 76.8 \text{ years}$$

To explain realistic cross-country differences in income per person as arising from slow transmission of knowledge to poor countries, we need the transmission to be very slow. Poor countries need to be using technology that the rich countries developed in the early 1900's if we are to explain a 10-fold difference in income per person.

b) i) Recall that in the Solow model, the balanced-growth-path value of $k \equiv K/AL$ is defined implicitly by the condition that actual investment $- sf(k^*)$ equal break-even investment $- (n + g + \delta)k^*$. Thus for the north, k_N^* is implicitly defined by:

$$(6) sf(k_N^*) = (n + g + \delta)k_N^*$$

where $g = \dot{A}_N(t)/A_N(t)$.

We are told that s, n, δ and the function $f(\bullet)$ are the same for the north and the south. The only possible source of difference is the growth rate of southern knowledge. However, it is straightforward to show that $\dot{A}_S(t)/A_S(t) = g$ also.

We are told that the knowledge used in the south at time t is the knowledge that was used in the north at time $t - \tau$. That is:

$$(7) A_S(t) = A_N(t - \tau)$$

Taking the time derivative of equation (7) yields:

$$(8) \dot{A}_S(t) = \dot{A}_N(t - \tau)$$

Dividing equation (8) by equation (7) yields:

$$(9) \frac{\dot{A}_S(t)}{A_S(t)} = \frac{\dot{A}_N(t - \tau)}{A_N(t - \tau)}$$

The growth rate of northern knowledge is constant and equal to g at all points in time and thus:

$$(10) \dot{A}_S(t)/A_S(t) = g$$

Therefore, for the south, k_S^* is implicitly defined by:

$$(11) sf(k_S^*) = (n + g + \delta)k_S^*$$

Since k_N^* and k_S^* are implicitly defined by the same parameters, they must be equal.

b) ii) Introducing capital will not change the answer to part a. Since $k_N^* = k_S^*$, output per unit of effective labor on the balanced growth path will also be equal in the north and the south. That is, $y_N^* = y_S^*$ where $y_i^* = [Y_i / A_i L_i]^*$. We can write the balanced-growth-path value of output per worker in the north as:

$$(12) Y_N(t)/L_N(t) \equiv A_N(t) \cdot y_N^*$$

Similarly, the balanced-growth-path value of output per worker in the south is:

$$(13) Y_S(t)/L_S(t) \equiv A_S(t) \cdot y_S^*$$

Dividing equation (12) by equation (13) yields:

$$(14) \frac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \frac{A_N(t) \cdot y_N^*}{A_S(t) \cdot y_S^*} = \frac{A_N(t)}{A_S(t)} = \frac{A_N(t)}{A_N(t - \tau)}$$

The second to last step uses the fact that $y_N^* = y_s^*$. The last step uses $A_s(t) = A_N(t - \tau)$. Using equation (3), we again have:

$$\frac{Y_N(t)/L_N(t)}{Y_S(t)/L_S(t)} = \frac{A_N(t)}{A_N(t - \tau)} = e^{0.03\tau}$$

The same calculation as in part a would yield a value of $\tau = 76.8$ years in order for $[Y_N(t)/L_N(t)]/[Y_S(t)/L_S(t)] = 10$.

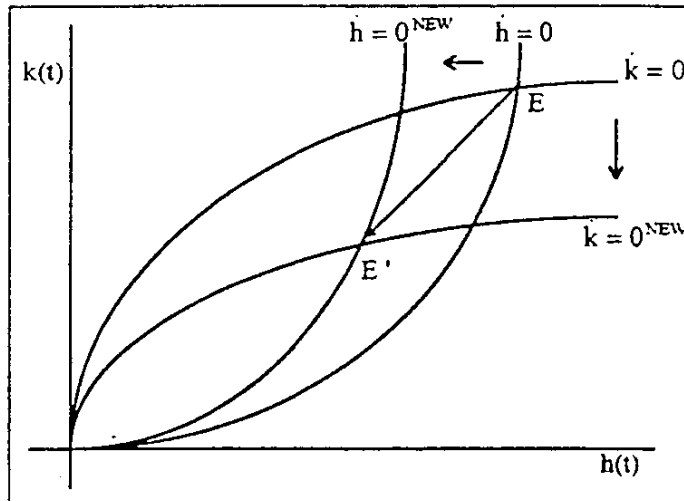
Problem 3.12

The equations of the $\dot{k} = 0$ and $\dot{h} = 0$ loci are:

$$(1) \quad k(t) = [s_K/(n+g)]^{1/(1-\alpha)} h(t)^{\beta/(1-\alpha)} \quad \dot{k} = 0 \quad (2) \quad k(t) = [(n+g)/s_H]^{1/\alpha} h(t)^{(1-\beta)/\alpha} \quad \dot{h} = 0$$

From equation (1), a rise in n shifts down the $\dot{k} = 0$ locus. From equation (2), the rise in n shifts up the $\dot{h} = 0$ locus. At point E, the economy is now above the new $\dot{k} = 0$ locus and so the amount of physical capital per unit of effective labor, $k(t)$, is falling.

Similarly, since at E, the economy is to the right of the new $\dot{h} = 0$ locus, the amount of human capital per unit of effective labor, $h(t)$, is falling. The economy moves down and left in (h, k) space until it reaches the new balanced growth path at point E'.



By definition, output per worker can be written as $Y/L \equiv Ay$ where y is output per unit of effective labor. Substituting in for the intensive form of the production function gives $Y/L = Ak^\alpha h^\beta$. On the initial balanced growth path, k and h are constant and so output per worker grows at rate $\dot{A}/A \equiv g$. During the transition between the two balanced growth paths, k and h are falling and thus output per worker grows at a rate less than g . When the economy reaches the new balanced growth path, k and h are again constant, and so the growth rate of output per worker returns to g . Thus the permanent decrease in the population growth rate leads only to a temporary decrease in the growth rate of output per worker. The path of output per worker will lie below the path it otherwise would have taken, however. These qualitative implications are essentially identical to those of the Solow model.

Problem 3.13

a) The balanced-growth-path value of consumption per unit of effective labor is given by:

$$(1) \quad c^* = (1 - s_K - s_H)y^*$$

Taking the exponential function of both sides of equation (3.57) in the text yields an expression for output per unit of effective labor on the balanced growth path:

$$(2) \quad y^* = [s_K^\alpha s_H^\beta / (n+g)^{\alpha+\beta}]^{1/(1-\alpha-\beta)}$$

Substituting equation (2) into equation (1) gives us c^* in terms of the underlying parameters of the model:

$$(3) \quad c^* = (1 - s_K - s_H) \cdot [s_K^\alpha s_H^\beta / (n+g)^{\alpha+\beta}]^{1/(1-\alpha-\beta)}$$

b) It will turn out to be easiest if we maximize the log of c^* with respect to s_K and s_H . Taking the log of both sides of equation (3) yields:

$$(4) \ln c^* = \ln(1 - s_K - s_H) + \frac{\alpha}{1 - \alpha - \beta} \ln s_K + \frac{\beta}{1 - \alpha - \beta} \ln s_H - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln(n + g)$$

The first-order conditions are:

$$(5) \frac{\partial \ln c^*}{\partial s_K} = \frac{-1}{1 - s_K - s_H} + \frac{\alpha}{1 - \alpha - \beta} \cdot \frac{1}{s_K} = 0 \quad \text{and} \quad (6) \frac{\partial \ln c^*}{\partial s_H} = \frac{-1}{1 - s_K - s_H} + \frac{\beta}{1 - \alpha - \beta} \cdot \frac{1}{s_H} = 0$$

Subtract equation (6) from equation (5) to obtain:

$$\frac{\alpha}{1 - \alpha - \beta} \cdot \frac{1}{s_K} - \frac{\beta}{1 - \alpha - \beta} \cdot \frac{1}{s_H} = 0 \quad \Rightarrow \quad \frac{\alpha}{1 - \alpha - \beta} \cdot \frac{1}{s_K} = \frac{\beta}{1 - \alpha - \beta} \cdot \frac{1}{s_H}$$

Simplifying and solving for s_H as a function of s_K gives us:

$$(7) s_H = (\beta/\alpha) \cdot s_K$$

Substitute equation (7) into equation (5):

$$\frac{-1}{1 - s_K - (\beta/\alpha)s_K} + \frac{\alpha}{1 - \alpha - \beta} \cdot \frac{1}{s_K} = 0 \quad \Rightarrow \quad \frac{\alpha}{1 - \alpha - \beta} \cdot \frac{1}{s_K} = \frac{\alpha}{\alpha - \alpha s_K - \beta s_K}$$

"Cross-multiplying" to simplify yields:

$$\alpha - \alpha s_K - \beta s_K = s_K - \alpha s_K - \beta s_K$$

and thus:

$$(8) s_K^* = \alpha$$

Substituting equation (8) back into equation (7) gives us $s_H = (\beta/\alpha) \cdot \alpha$ or simply:

$$(9) s_H^* = \beta$$

In order for consumption per unit of effective labor to be maximized on the balanced growth path, the saving rate for physical capital must equal physical capital's share in output — $s_K^* = \alpha$ — and the fraction of resources devoted to human capital accumulation must equal human capital's share in output — $s_H^* = \beta$. Note that this is analogous to the result for the Solow model with a Cobb-Douglas production function. In that case, c^* was maximized when the saving rate equaled capital's share in output. See Problem 1.2.

Problem 3.14

a) From equation (3.57) in the text, the log of output per unit of effective labor on the balanced growth path is:

$$(1) \ln y^* = \frac{\alpha}{1 - \alpha - \beta} \ln s_K + \frac{\beta}{1 - \alpha - \beta} \ln s_H - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln(n + g)$$

Thus the elasticity of output per unit of effective labor with respect to the saving rate for physical capital is given by:

$$\frac{\partial y^*}{\partial s_K} \cdot \frac{s_K}{y^*} \equiv \frac{\partial \ln y^*}{\partial \ln s_K} = \frac{\alpha}{1 - \alpha - \beta} = \frac{0.35}{1 - 0.35 - 0.4} = 1.4$$

where we have used the values given for α and β . A rise in s_K from $s_K = 0.15$ to $s_K^{\text{NEW}} = 0.18$ — which is what occurs, since we are assuming that the entire 3% of output is saved after the deficit is eliminated — is a 20% increase in the saving rate for physical capital. Using the value of 1.4 for the elasticity, this means that output per unit of effective labor will rise by about $(0.20) \cdot (1.4) = 0.28$ or 28%. Since the paths of A and L are unaffected, this also means that output itself will be 28% higher than it would have been without the deficit reduction.

b) Consumption per unit of effective labor on the balanced growth path is given by:

$$(2) c^* = (1 - s_K - s_H) y^*$$

Taking the log of both sides of equation (2) gives us:

$$(3) \ln c^* = \ln(1 - s_K - s_H) + \ln y^*$$

Using equation (1) to substitute for $\ln y^*$ yields:

$$(4) \ln c^* = \ln(1 - s_K - s_H) + \frac{\alpha}{1 - \alpha - \beta} \ln s_K + \frac{\beta}{1 - \alpha - \beta} \ln s_H - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln(n + g)$$

The difference between log consumption per unit of effective labor on the new balanced growth path with s_K^{new} and that with s_K is given by:

$$\ln c^*_{\text{new}} - \ln c^* = \ln(1 - s_K^{\text{new}} - s_H) - \ln(1 - s_K - s_H) + \frac{\alpha}{1 - \alpha - \beta} [\ln s_K^{\text{new}} - \ln s_K]$$

Substituting in for $s_K^{\text{new}} = 0.18$, $s_K = 0.15$, $s_H = 0.15$, $\alpha = 0.35$ and $\beta = 0.4$ yields:

$$\ln c^*_{\text{new}} - \ln c^* = \ln(1 - 0.18 - 0.15) - \ln(1 - 0.15 - 0.15) + \frac{0.35}{1 - 0.35 - 0.4} [\ln(0.18) - \ln(0.15)]$$

or simply:

$$\ln c^*_{\text{new}} - \ln c^* \approx 0.2114$$

Since $e^{0.2114} \approx 1.2355$, consumption is approximately 23.55% higher than it would have been without the deficit reduction.

c) The immediate effect of the deficit reduction is that consumption falls. Although y^* does not jump immediately — it only begins to move gradually toward its new, higher balanced-growth-path level — we are now saving a greater fraction and thus consuming a smaller fraction of this same y^* . At the moment of the rise in s_K by 3 percentage points — since $c^* = (1 - s_K - s_H)y^*$ and y^* is unchanged — c falls. In fact, the percentage change in c will be the percentage change in $(1 - s_K - s_H)$. Now, $(1 - s_K - s_H)$ falls from 0.7 to 0.67, which is approximately a 4.29% drop — that is, 0.03 is about 4.29% of 0.7. Thus at the moment of the rise in s_K , consumption falls by about 4.29%.

We can use some results from Problem 3.19 on the speed of convergence in this model to determine how long it takes for consumption to return to what it would have been in the absence of the deficit reduction. After the initial rise in s_K , s_K remains constant throughout. Since $c = (1 - s_K - s_H)y$, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In Problem 3.19, it is shown that the rate of convergence of y — after a linear approximation — is given by $\lambda = (1 - \alpha - \beta)(n + g)$. With $(n + g) = 6\%$, $\alpha = 0.35$ and $\beta = 0.40$, $\lambda = 1.5\%$. This means that y moves 1.5% of the remaining distance toward the balanced-growth-path value of y^* each year. Since c is proportional to y , it also approaches its new balanced-growth-path value at this same constant rate. Thus we can write:

$$(5) c(t) - c^* \approx e^{-(1 - \alpha - \beta)(n + g)t} [c(0) - c^*]$$

where $c(0)$ is the initial value of c at the moment that s_K changes. Rewriting (5):

$$(6) e^{-\lambda t} = \frac{c(t) - c^*}{c(0) - c^*}$$

The term on the right-hand side of equation (6) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 4.29% and will eventually be 23.55% higher than it would have been. Thus it must change by a total of 27.84% on the way to the balanced growth path. It will therefore be equal to what it would have been, $4.29\%/27.84\% \approx 15.4\%$ of the way to the new balanced growth path. Equivalently, this is when the remaining distance to the new balanced growth path is 84.6% of the original distance. In order to find out how long this will take, we need to find a t^* that solves:

$$(7) e^{-\lambda t^*} = 0.846$$

Taking the natural log of both sides of equation (7) yields:

$$-\lambda \cdot t^* = \ln(0.846) \Rightarrow t^* = 0.167/0.015$$

and thus:

$$(8) t^* \approx 11.1 \text{ years}$$

It will take a fairly long time — over a decade — for consumption to return to what it would have been in the absence of the deficit reduction.

d) Compared to the results for the Solow model in Problem 1.6, output and consumption rise much more in this model with human capital. Output rises by 28% as compared to 10%, while consumption rises by about 23.55% as compared to 6%. Thus introducing human capital into the model makes the benefits of deficit reduction much more attractive. However, the rate of convergence in this model is much slower than in the Solow model. This offsets the fact that consumption returns to its original value earlier in the transition to the new balanced growth path in this model. The end result is that in both models, the time it takes for consumption to return to what it would have been in the absence of the deficit reduction is quite long — over 11 years.

Problem 3.15

a) Differentiating both sides of the definition of $k(t) \equiv K(t)/A(t)L(t)$ with respect to time yields:

$$(1) \dot{k}(t) = \frac{\dot{K}(t)A(t)L(t) - K(t)[\dot{A}(t)L(t) + A(t)\dot{L}(t)]}{[A(t)L(t)]^2}$$

Using the definition of $k(t) \equiv K(t)/A(t)L(t)$, equation (1) can be rewritten as:

$$(2) \dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - \left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} \right] k(t)$$

Substituting the capital accumulation equation — $\dot{K}(t) = sY(t) - \delta_K K(t)$ — as well as the constant growth rates of knowledge and labor into equation (2) gives us:

$$(3) \dot{k}(t) = \frac{sY(t) - \delta_K K(t)}{A(t)L(t)} - (n + g)k(t)$$

Substituting the production function — $Y(t) = [(1 - a_K)K(t)]^\alpha [(1 - a_H)H(t)]^{1-\alpha}$ — into equation (3) yields:

$$(4) \dot{k}(t) = s \left[\frac{(1 - a_K)K(t)}{A(t)L(t)} \right]^\alpha \left[\frac{(1 - a_H)H(t)}{A(t)L(t)} \right]^{1-\alpha} - (n + g + \delta_K)k(t)$$

Finally, defining $c_K \equiv s(1 - a_K)^\alpha (1 - a_H)^{1-\alpha}$ and using $k(t) \equiv K(t)/A(t)L(t)$ as well as $h(t) \equiv H(t)/A(t)L(t)$, equation (4) can be rewritten as:

$$(5) \dot{k}(t) = c_K k(t)^\alpha h(t)^{1-\alpha} - (n + g + \delta_K)k(t)$$

Differentiating both sides of the definition of $h(t) \equiv H(t)/A(t)L(t)$ with respect to time yields:

$$(6) \dot{h}(t) = \frac{\dot{H}(t)A(t)L(t) - H(t)[\dot{A}(t)L(t) + A(t)\dot{L}(t)]}{[A(t)L(t)]^2}$$

Equation (6) can be simplified to:

$$(7) \dot{h}(t) = \frac{\dot{H}(t)}{A(t)L(t)} - \left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} \right] h(t)$$

Substituting $\dot{H}(t) = B[a_K K(t)]^\gamma [a_H H(t)]^\phi [A(t)L(t)]^{1-\gamma-\phi} - \delta_H H(t)$, the human capital accumulation equation, as well as the constant growth rates of knowledge and labor into equation (7) gives us:

$$(8) \dot{h}(t) = B \left[\frac{a_K K(t)}{A(t)L(t)} \right]^\gamma \left[\frac{a_H H(t)}{A(t)L(t)} \right]^\phi \left[\frac{A(t)L(t)}{A(t)L(t)} \right]^{1-\gamma-\phi} - (n+g+\delta_H)k(t)$$

Finally, defining $c_H \equiv Ba_K^\gamma a_H^\phi$ allows us to rewrite equation (8) as:

$$(9) \dot{h}(t) = c_H k(t)^\gamma h(t)^\phi - (n+g+\delta_H)h(t)$$

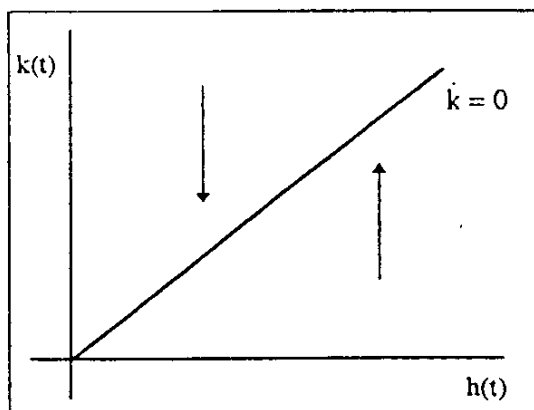
b) To find the combinations of h and k such that $\dot{k} = 0$, set the right-hand side of equation (5) equal to zero and solve for k as a function of h :

$$c_K k(t)^\alpha h(t)^{1-\alpha} = (n+g+\delta_K)k(t) \Rightarrow k(t)^{1-\alpha} = c_K h(t)^{1-\alpha} / (n+g+\delta_K)$$

and thus finally:

$$(10) k(t) = [c_K / (n+g+\delta_K)]^{1/(1-\alpha)} h(t)$$

The $\dot{k} = 0$ locus, as defined by equation (10), is a straight line with slope $[c_K / (n+g+\delta_K)]^{1/(1-\alpha)} > 0$ that passes through the origin. See the diagram at right. From equation (5), we can see that $\dot{k}(t)$ is increasing in $h(t)$. Thus to the right of the $\dot{k} = 0$ locus, $\dot{k} > 0$ and so $k(t)$ is rising. To the left of the $\dot{k} = 0$ locus, $\dot{k} < 0$ and so $k(t)$ is falling.



To find the combinations of h and k such that $\dot{h} = 0$, set the right-hand side of equation (9) equal to zero and solve for k as a function of h .

$$c_H k(t)^\gamma h(t)^\phi = (n+g+\delta_H)h(t) \Rightarrow k(t)^\gamma = [(n+g+\delta_H)/c_H] h(t)^{1-\phi}$$

and thus finally:

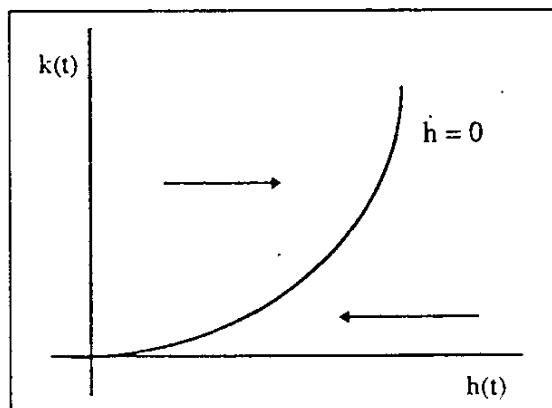
$$(11) k(t) = [c_H / (n+g+\delta_H)]^{1/\gamma} h(t)^{(1-\phi)/\gamma}$$

The following derivatives will be useful:

$$dk(t)/dh(t)|_{h=0} = [(1-\phi)/\gamma] \cdot [c_H / (n+g+\delta_H)]^{1/\gamma} h(t)^{(1-\phi)/\gamma - 1} > 0$$

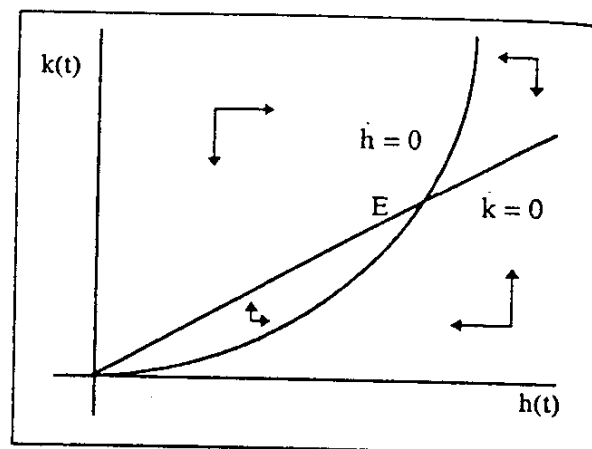
$$d^2 k(t)/dh(t)^2|_{h=0} = [(1-\phi-\gamma)/\gamma] \cdot [(1-\phi)/\gamma] \cdot [c_H / (n+g+\delta_H)]^{1/\gamma} h(t)^{(1-\phi)/\gamma - 2} > 0$$

The $\dot{h} = 0$ locus, as defined by equation (11), is upward sloping with a positive second derivative. See the diagram at right. From equation (9), we can see that $\dot{h}(t)$ is increasing in $k(t)$. Therefore, above the $\dot{h} = 0$ locus, $\dot{h} > 0$ and so $h(t)$ is increasing. Below the $\dot{h} = 0$ locus, $\dot{h} < 0$ and so $h(t)$ is falling.



c) Putting the $\dot{k} = 0$ and $\dot{h} = 0$ loci together, we can see that the economy will converge to a stable balanced growth path at point E. This stable balanced growth path is unique (as long as we ignore the origin with $k = h = 0$).

From the diagram, physical capital per unit of effective labor -- $k(t) \equiv K(t)/A(t)L(t)$ -- is constant on a balanced growth path. Thus physical capital per person -- $K(t)/L(t) \equiv k(t)A(t)$ -- must grow at the same rate as knowledge, which is g . Similarly, human capital per unit of effective labor -- $h(t) \equiv H(t)/A(t)L(t)$ -- is constant on the balanced growth path. Thus human capital per person -- $H(t)/L(t) \equiv h(t)A(t)$ -- must also grow at the same rate as knowledge, which is g .

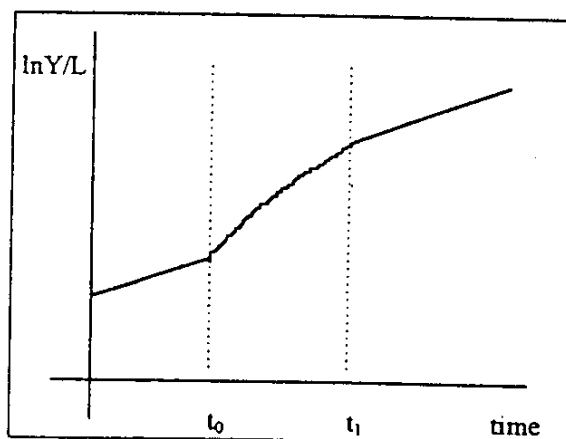
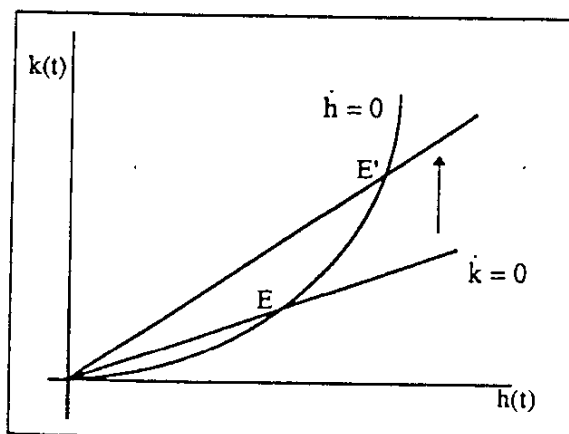


Dividing the production function by $L(t)$ gives us an expression for output per person:

$$(12) Y(t)/L(t) = [(1 - a_K)K(t)/L(t)]^\alpha [(1 - a_H)H(t)/L(t)]^{1-\alpha}$$

Since $K(t)/L(t)$ and $H(t)/L(t)$ both grow at rate g on the balanced growth path and since the production function is constant returns to scale, output per person also grows at rate g on the balanced growth path.

d) From equation (10), the slope of the $\dot{k} = 0$ locus is $[c_K / (n + g + \delta_K)]^{1/(1-\alpha)}$ where we have defined $c_K \equiv s(1 - a_K)^\alpha (1 - a_H)^{1-\alpha}$. Thus a rise in s will make the $\dot{k} = 0$ locus steeper. Since s does not appear in equation (11), the $\dot{h} = 0$ locus is unaffected. See the diagram on the left. The economy will move from its old balanced growth path at E to a new balanced growth path at E'.



Output per person grows at rate g until the time that s rises (denoted time t_0 in the diagram on the right). During the transition from E to E', both $h(t)$ and $k(t)$ are rising. Thus human capital per person and physical capital per person grow at a rate greater than g during the transition. From equation (12), this means that output per person grows at a rate greater than g during the transition as well. Once the economy reaches the new balanced growth path (at time t_1 in the diagram), $h(t)$ and $k(t)$ are constant again. Thus human and physical capital per person grow at rate g again. Thus output per person grows at rate g again on the new balanced growth path. A permanent rise in the saving rate has only a level effect on output per person, not a permanent growth rate effect.

Problem 3.16

The production function is given by:

$$(1) Y(t) = K(t)^\alpha H(t)^\beta [A(t)L(t)]^{1-\alpha-\beta}$$

The marginal product of physical capital is thus $MPK \equiv \partial Y(t)/\partial K(t) = \alpha K(t)^{\alpha-1} H(t)^\beta [A(t)L(t)]^{1-\alpha-\beta}$, which can be written as:

$$(2) MPK = \alpha [K(t)/A(t)L(t)]^{\alpha-1} [H(t)/A(t)L(t)]^\beta = \alpha k(t)^{\alpha-1} h(t)^\beta$$

where we have used $k(t) \equiv K(t)/A(t)L(t)$ and $h(t) \equiv H(t)/A(t)L(t)$.

Taking the exponential function of equations (3.55) and (3.56) in the text will yield expressions for k^* and h^* , the amounts of physical and human capital per unit of effective labor on the balanced growth path:

$$(3) k^* = [s_K^{1-\beta} s_H^\beta / (n+g)]^{1/(1-\alpha-\beta)} \quad (4) h^* = [s_K^\alpha s_H^{1-\alpha} / (n+g)]^{1/(1-\alpha-\beta)}$$

Substitute equations (3) and (4) into equation (2):

$$MPK^* = \alpha [s_K^{1-\beta} s_H^\beta / (n+g)]^{(\alpha-1)/(1-\alpha-\beta)} [s_K^\alpha s_H^{1-\alpha} / (n+g)]^{\beta/(1-\alpha-\beta)}$$

Now collect and simplify the exponents:

$$MPK^* = \alpha [s_K^{(1-\beta)(\alpha-1)+\alpha\beta} s_H^{\beta(\alpha-1)+(1-\alpha)\beta} / (n+g)^{(\alpha-1)+\beta}]^{1/(1-\alpha-\beta)}$$

$$MPK^* = \alpha [s_K^{-(1-\alpha-\beta)} s_H^0 / (n+g)^{-(1-\alpha-\beta)}]^{1/(1-\alpha-\beta)}$$

or simply:

$$(5) MPK^* = \alpha(n+g)/s_K$$

The marginal product of human capital is $MPH \equiv \partial Y(t)/\partial H(t) = \beta K(t)^\alpha H(t)^{\beta-1} [A(t)L(t)]^{1-\alpha-\beta}$, which can be written as:

$$(6) MPH = \beta [K(t)/A(t)L(t)]^\alpha [H(t)/A(t)L(t)]^{\beta-1} = \beta k(t)^\alpha h(t)^{\beta-1}$$

Substitute equations (3) and (4) into equation (6):

$$MPH^* = \beta [s_K^{1-\beta} s_H^\beta / (n+g)]^{\alpha/(1-\alpha-\beta)} [s_K^\alpha s_H^{1-\alpha} / (n+g)]^{(\beta-1)/(1-\alpha-\beta)}$$

Now collect and simplify the exponents:

$$MPH^* = \beta [s_K^{\alpha(1-\beta)+\alpha(\beta-1)} s_H^{\alpha\beta+(1-\alpha)(\beta-1)} / (n+g)^{\alpha+(\beta-1)}]^{1/(1-\alpha-\beta)}$$

$$MPH^* = \beta [s_K^0 s_H^{-(1-\alpha-\beta)} / (n+g)^{-(1-\alpha-\beta)}]^{1/(1-\alpha-\beta)}$$

or simply:

$$(7) MPH^* = \beta(n+g)/s_H$$

Problem 3.17

a) Taking the time derivative of the ratio of physical capital to human capital yields:

$$(1) [K(t)/H(t)] = \frac{1}{H(t)^2} [\dot{K}(t)H(t) - K(t)\dot{H}(t)]$$

Substituting the accumulation equations for physical and human capital — $\dot{K}(t) = s_K Y(t)$ and $\dot{H}(t) = s_H Y(t)$ — into equation (1) gives us:

$$(2) [K(t)/H(t)] = \frac{1}{H(t)^2} [s_K Y(t)H(t) - s_H Y(t)K(t)]$$

which can be rearranged to yield:

$$(3) \left[\dot{K}(t) / H(t) \right] = \frac{Y(t)}{H(t)} \left[s_K - s_H \left(\frac{K(t)}{H(t)} \right) \right]$$

From equation (3), we can see that K/H will be rising when $s_K - s_H (K/H) > 0$ or when $K/H < s_K / s_H$. Similarly, K/H will be falling when $s_K - s_H (K/H) < 0$ or when $K/H > s_K / s_H$. Thus K/H does converge to a balanced-growth-path level of $(K/H)^* = s_K / s_H$.

b) Divide both sides of the physical capital accumulation equation by $K(t)$ to yield:

$$(4) \frac{\dot{K}(t)}{K(t)} = \frac{s_K Y(t)}{K(t)}$$

Substituting the production function — $Y(t) = K(t)^\alpha H(t)^{1-\alpha}$ — into equation (4) gives us:

$$(5) \frac{\dot{K}(t)}{K(t)} = \frac{s_K K(t)^\alpha H(t)^{1-\alpha}}{K(t)} = s_K \left[\frac{H(t)}{K(t)} \right]^{1-\alpha}$$

From part (a), we know that on the balanced growth path, $(K/H)^* = s_K / s_H$. Substituting this fact into equation (5) gives us the growth rate of K on the balanced growth path:

$$(6) \frac{\dot{K}(t)}{K(t)} = s_K \left[\frac{s_H}{s_K} \right]^{1-\alpha} = s_K^\alpha s_H^{1-\alpha}$$

On the balanced growth path, K/H is constant and thus H must be growing at the same rate as K . Finally, since the production function has constant returns to scale and K and H are both growing at the same rate, output must also grow at rate $s_K^\alpha s_H^{1-\alpha}$ on the balanced growth path.

c) Since the growth rate of output on the balanced growth path is $s_K^\alpha s_H^{1-\alpha}$, an increase in either s_K or s_H will permanently raise the growth rate of output. This is different than in the standard human capital model in the text with diminishing returns to just physical and human capital. In that case, we obtained Solow type results that a rise in the saving rate had a "level effect" but not a permanent "growth rate effect" on the economy.

d) Taking the time derivative of the log of the production function yields an expression for the growth rate of output:

$$(7) g_Y = \alpha g_K + (1 - \alpha) g_H$$

where g_K and g_H denote the growth rates of physical and human capital. Equation (5) gives us an expression for g_K . We need a similar expression for g_H . Dividing both sides of the human capital accumulation equation by $H(t)$ yields:

$$(8) \frac{\dot{H}(t)}{H(t)} = \frac{s_H K(t)^\alpha H(t)^{1-\alpha}}{H(t)} = s_H \left[\frac{K(t)}{H(t)} \right]^\alpha$$

Substituting equations (5) and (8) into equation (7) gives us:

$$(9) g_Y = \alpha s_K \left(\frac{K}{H} \right)^{\alpha-1} + (1 - \alpha) s_H \left(\frac{K}{H} \right)^\alpha$$

We can now examine how the growth rate of output changes when the ratio of physical to human capital changes. Use equation (9) to take the derivative of the growth rate of output with respect to the ratio of physical to human capital:

$$\partial g_Y / \partial (K/H) = \alpha(\alpha - 1) s_K \left(\frac{K}{H} \right)^{\alpha-2} + \alpha(1 - \alpha) s_H \left(\frac{K}{H} \right)^{\alpha-1}$$

Factoring out $\alpha(1 - \alpha) s_H \left(\frac{K}{H} \right)^{\alpha-2}$ from each term yields:

$$(10) \frac{\partial g_Y}{\partial (K/H)} = \alpha(1-\alpha)s_H(K/H)^{\alpha-2} \left[\frac{K}{H} - \frac{s_K}{s_H} \right]$$

Clearly, $\alpha(1-\alpha)s_H(K/H)^{\alpha-2} > 0$. Thus if K/H is less than its balanced-growth-path level of s_K/s_H we have $\partial g_Y/\partial (K/H) < 0$. Thus when K/H starts below its balanced-growth-path level and then rises towards it, the growth rate of output must be falling the entire time that $K/H < s_K/s_H$. If the growth rate of output is falling throughout and will be equal to g^* once K/H reaches $(K/H)^*$, it must be greater than g^* initially. That is, if K/H starts out at a level smaller than its eventual balanced-growth-path value, the initial growth rate of output will be greater than its growth rate on the balanced growth path.

Problem 3.18

The relevant equations are:

$$(1) Y(t) = K(t)^\alpha [(1-a_H)H(t)]^\beta \quad \alpha + \beta > 1 \quad (2) \dot{H}(t) = Ba_H H(t) \quad (3) \dot{K}(t) = sY(t)$$

a) To get the growth rate of human capital — which turns out to be constant — divide equation (2) by $H(t)$:

$$(4) g_H \equiv \dot{H}(t)/H(t) = Ba_H$$

b) Substitute the production function, equation (1), into the expression for the evolution of the physical capital stock, equation (3):

$$(5) \dot{K}(t) = sK(t)^\alpha [(1-a_H)H(t)]^\beta$$

To get the growth rate of physical capital, divide equation (5) by $K(t)$:

$$(6) g_K(t) \equiv \dot{K}(t)/K(t) = sK(t)^{\alpha-1} [(1-a_H)H(t)]^\beta$$

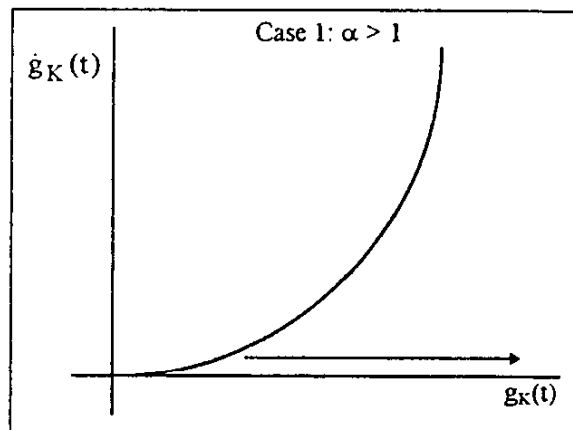
We need to examine the dynamics of the growth rate of physical capital. Taking the time derivative of the log of equation (6) yields the growth rate of the growth rate of physical capital:

$$(7) \dot{g}_K(t)/g_K(t) = (\alpha-1) \cdot \dot{K}(t)/K(t) + \beta \cdot \dot{H}(t)/H(t) = (\alpha-1)g_K(t) + \beta g_H$$

Now plot the change in the growth rate of capital, $\dot{g}_K(t)$, as a function of the growth rate of capital itself, $g_K(t)$ — where, from equation (7):

$$(8) \dot{g}_K(t) = (\alpha-1)g_K(t)^2 + \beta g_H g_K(t)$$

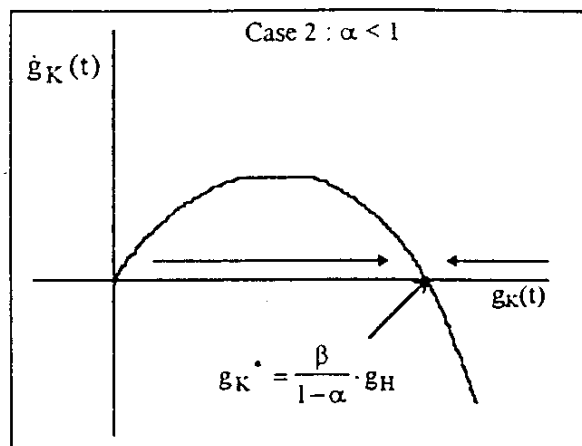
It turns out that whether or not the economy converges to a balanced growth path depends on whether α is greater or less than one — that is, whether there are increasing or decreasing returns to physical capital alone. The case of $\alpha > 1$ — increasing returns to physical capital alone — is depicted at right. In this case, equation (8) is a parabola with a root at $g_K(t) = 0$. Regardless of the initial growth rate of capital, as long as it is not 0, the growth rate of capital increases without bound. Thus the economy does not converge to a balanced growth path. Note that the case of $\alpha = 1$ — constant returns to physical capital alone — is similar. In that case, $\dot{g}_K(t)$ is a linear function of $g_K(t)$ but the implication is the same — $g_K(t)$ increases without bound and there is no balanced growth path.



Now consider the case of $\alpha < 1$ — decreasing returns to physical capital alone. The phase diagram is depicted at right. Equation (8) now describes an "inverted" parabola. Now, $g_K(t)$ is constant when $\dot{g}_K(t) = 0$ or when:

$(\alpha - 1)g_K(t) + \beta g_H = 0$. Solving for $g_K(t)$ yields:
 $g_K^* = [\beta/(1 - \alpha)]g_H$.

Note that $g_K^* > g_H$ since $\alpha + \beta > 1$ or $\beta > 1 - \alpha$. To the left of g_K^* , from the phase diagram, $\dot{g}_K(t) > 0$ and so $g_K(t)$ rises towards g_K^* .



Similarly, to the right of g_K^* , $\dot{g}_K(t) < 0$ and so $g_K(t)$ falls towards g_K^* . Thus, if $\alpha < 1$, the growth rate of capital converges to a constant value of g_K^* and a balanced growth path exists.

Taking the time derivative of the log of equation (1) yields the growth rate of output:

$$(9) \quad \dot{Y}(t)/Y(t) = \alpha \cdot \dot{K}(t)/K(t) + \beta \cdot \dot{H}(t)/H(t) = \alpha g_K(t) + \beta g_H$$

On the balanced growth path, $g_K(t) = g_K^* = [\beta/(1 - \alpha)]g_H$ and so:

$$(10) \quad \frac{\dot{Y}(t)}{Y(t)} = \frac{\alpha\beta}{(1-\alpha)}g_H + \beta g_H = \frac{\alpha\beta + \beta - \alpha\beta}{(1-\alpha)}g_H = \frac{\beta}{(1-\alpha)}g_H \equiv g_K^*$$

On the balanced growth path, output grows at the same rate as physical capital, which in turn is greater than the constant growth rate of human capital, $g_H = \beta a_H$.

Problem 3.19

a) Taking the log of the production function — $y(t) = k(t)^\alpha h(t)^\beta$ — yields:

$$(1) \quad \ln y(t) = \alpha \ln k(t) + \beta \ln h(t)$$

Taking the time derivative of equation (1) gives us:

$$(2) \quad d \ln y(t)/dt = \alpha \cdot \dot{k}(t)/k(t) + \beta \cdot \dot{h}(t)/h(t)$$

Substituting into equation (2) the expressions for the dynamics of physical and human capital per unit of effective labor — $\dot{k}(t) = s_K k(t)^\alpha h(t)^\beta - (n + g)k(t)$ and $\dot{h}(t) = s_H k(t)^\alpha h(t)^\beta - (n + g)h(t)$ — yields:

$$(3) \quad d \ln y(t)/dt = \alpha [s_K k(t)^{\alpha-1} h(t)^\beta - (n + g)] + \beta [s_H k(t)^\alpha h(t)^{\beta-1} - (n + g)]$$

b) The first-order Taylor approximation will be of the form:

$$(4) \quad \frac{d \ln y(t)}{dt} \cong \left[\frac{\partial (d \ln y(t)/dt)}{\partial \ln k(t)} \right] \cdot [\ln k(t) - \ln k^*] + \left[\frac{\partial (d \ln y(t)/dt)}{\partial \ln h(t)} \right] \cdot [\ln h(t) - \ln h^*]$$

where the derivatives are evaluated at the balanced-growth-path values, $\ln k = \ln k^*$ and $\ln h = \ln h^*$. Using equation (3) to find the first derivative, we have:

$$\frac{\partial (d \ln y(t)/dt)}{\partial \ln k(t)} \equiv k(t) \cdot \frac{\partial (d \ln y(t)/dt)}{\partial k(t)} = k(t) \cdot [\alpha s_K (\alpha - 1) k(t)^{\alpha-2} h(t)^\beta + \beta s_H \alpha k(t)^{\alpha-1} h(t)^{\beta-1}]$$

or simply:

$$(5) \quad \frac{\partial (d \ln y(t)/dt)}{\partial \ln k(t)} = \alpha s_K (\alpha - 1) k(t)^{\alpha-1} h(t)^\beta + \beta s_H \alpha k(t)^\alpha h(t)^{\beta-1}$$

From equations (3.55) and (3.56) in the text, the balanced-growth-path values of physical and human capital per unit of effective labor are:

$$(6) \quad k^* = \left[\frac{s_K^{1-\beta} s_H^\beta}{(n+g)} \right]^{1/(1-\alpha-\beta)} \quad \text{and} \quad (7) \quad h^* = \left[\frac{s_K^\alpha s_H^{1-\alpha}}{(n+g)} \right]^{1/(1-\alpha-\beta)}$$

Substituting equations (6) and (7) into equation (5) yields:

$$\frac{\partial [d \ln y(t)/dt]}{\partial \ln k(t)} = \alpha s_K (\alpha - 1) \left[\frac{s_K^{1-\beta} s_H^\beta}{(n+g)} \right]^{(\alpha-1)/(1-\alpha-\beta)} \left[\frac{s_K^\alpha s_H^{1-\alpha}}{(n+g)} \right]^{\beta/(1-\alpha-\beta)} +$$

$$\beta s_H \alpha \left[\frac{s_K^{1-\beta} s_H^\beta}{(n+g)} \right]^{\alpha/(1-\alpha-\beta)} \left[\frac{s_K^\alpha s_H^{1-\alpha}}{(n+g)} \right]^{(\beta-1)/(1-\alpha-\beta)}$$

With some tedious manipulation of the exponents, this expression simplifies greatly:

$$\frac{\partial [d \ln y(t)/dt]}{\partial \ln k(t)} = \alpha(\alpha - 1) \left[s_K^{1-\alpha-\beta+\alpha-1-\alpha\beta+\beta+\alpha\beta} s_H^{\alpha\beta-\beta+\beta-\alpha\beta} (n+g)^{-\alpha+1-\beta} \right]^{1/(1-\alpha-\beta)} +$$

$$\beta \alpha \left[s_K^{\alpha-\alpha\beta+\alpha\beta-\alpha} s_H^{1-\alpha-\beta+\alpha\beta+\beta-1-\alpha\beta+\alpha} (n+g)^{-\alpha-\beta+1} \right]^{1/(1-\alpha-\beta)}$$

or simply:

$$\frac{\partial [d \ln y(t)/dt]}{\partial \ln k(t)} = \alpha(\alpha - 1)(n+g) + \beta \alpha(n+g) = \alpha(n+g)(\alpha - 1 + \beta)$$

and thus finally:

$$(8) \quad \frac{\partial [d \ln y(t)/dt]}{\partial \ln k(t)} = -\alpha(1 - \alpha - \beta)(n+g)$$

Now use equation (3) to take the other derivative required for the Taylor approximation:

$$\frac{\partial [d \ln y(t)/dt]}{\partial \ln h(t)} = h(t) \cdot \frac{\partial [d \ln y(t)/dt]}{\partial h(t)} = h(t) \cdot \left[\alpha \beta s_K k(t)^{\alpha-1} h(t)^{\beta-1} + \beta(\beta - 1) s_H k(t)^\alpha h(t)^{\beta-2} \right]$$

or simply:

$$(9) \quad \frac{\partial [d \ln y(t)/dt]}{\partial \ln h(t)} = \alpha \beta s_K k(t)^{\alpha-1} h(t)^\beta + \beta(\beta - 1) s_H k(t)^\alpha h(t)^{\beta-1}$$

Substitute equations (6) and (7) -- the balanced-growth-path values of physical and human capital per unit of effective labor -- into equation (9) in order to evaluate the derivative at the balanced growth path:

$$\frac{\partial [d \ln y(t)/dt]}{\partial \ln h(t)} = \alpha \beta s_K \left[\frac{s_K^{1-\beta} s_H^\beta}{(n+g)} \right]^{(\alpha-1)/(1-\alpha-\beta)} \left[\frac{s_K^\alpha s_H^{1-\alpha}}{(n+g)} \right]^{\beta/(1-\alpha-\beta)} +$$

$$\beta(\beta - 1) s_H \left[\frac{s_K^{1-\beta} s_H^\beta}{(n+g)} \right]^{\alpha/(1-\alpha-\beta)} \left[\frac{s_K^\alpha s_H^{1-\alpha}}{(n+g)} \right]^{(\beta-1)/(1-\alpha-\beta)}$$

Again, some tedious manipulation of the exponents simplifies the expression greatly. Note that the required manipulation is exactly the same as above and thus is not repeated here. The required simplification yields:

$$\frac{\partial [d \ln y(t)/dt]}{\partial \ln h(t)} = \alpha \beta(n+g) + \beta(\beta - 1)(n+g) = \beta(n+g)(\alpha + \beta - 1)$$

and thus finally:

$$(10) \quad \frac{\partial [d \ln y(t)/dt]}{\partial \ln h(t)} = -\beta(1 - \alpha - \beta)(n + g)$$

Substituting equations (8) and (10) into the Taylor approximation – equation (4) – yields:

$$(11) \quad d \ln y(t)/dt \cong -\alpha(1 - \alpha - \beta)(n + g)[\ln k(t) - \ln k^*] - \beta(1 - \alpha - \beta)(n + g)[\ln h(t) - \ln h^*]$$

c) Equation (11) can be rewritten as:

$$d \ln y(t)/dt \cong -(1 - \alpha - \beta)(n + g)[\alpha \ln k(t) + \beta \ln h(t) - (\alpha \ln k^* + \beta \ln h^*)]$$

From equation (1), $\ln y(t) = \alpha \ln k(t) + \beta \ln h(t)$ and on the balanced growth path, $\ln y^* = \alpha \ln k^* + \beta \ln h^*$.

Using these facts, the Taylor approximation can be rewritten as:

$$d \ln y(t)/dt \cong -(1 - \alpha - \beta)(n + g)[\ln y(t) - \ln y^*]$$

Defining the rate of convergence as $\lambda \equiv (1 - \alpha - \beta)(n + g)$, we have equation (3.65) in the text:

$$d \ln y(t)/dt \cong -\lambda[\ln y(t) - \ln y^*]$$

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SOLUTIONS TO CHAPTER 4

Problem 4.3

a) The equations describing the evolution of technology are given by:

$$(1) \ln A_t = \bar{A} + g + \tilde{A}_t \quad \text{and} \quad (2) \tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{A,t} \quad -1 < \rho_A < 1$$

From equation (1) and letting $\ln A_0$ denote the value of $\ln A$ in period 0, we have $\ln A_0 = \bar{A} + g(0) + \tilde{A}_0$.

Rearranging to solve for $\tilde{A}_0$ gives us:

$$(3) \tilde{A}_0 = \ln A_0 - \bar{A}$$

Now, in period 1 -- using equations (1) and (2) -- we have:

$$(4) \ln A_1 = \bar{A} + g + \tilde{A}_1 \quad \text{and} \quad (5) \tilde{A}_1 = \rho_A \tilde{A}_0 + \varepsilon_{A,1}$$

Substituting equation (3) into equation (5) yields:

$$(6) \tilde{A}_1 = \rho_A (\ln A_0 - \bar{A}) + \varepsilon_{A,1}$$

Finally, substituting equation (6) into equation (4) gives us:

$$(7) \ln A_1 = \bar{A} + g + \rho_A (\ln A_0 - \bar{A}) + \varepsilon_{A,1}$$

Now, in period 2 -- using equations (1) and (2) -- we have:

$$(8) \ln A_2 = \bar{A} + 2g + \tilde{A}_2 \quad \text{and} \quad (9) \tilde{A}_2 = \rho_A \tilde{A}_1 + \varepsilon_{A,2}$$

Substituting equation (6) into equation (9) yields:

$$(10) \tilde{A}_2 = \rho_A [\rho_A (\ln A_0 - \bar{A}) + \varepsilon_{A,1}] + \varepsilon_{A,2} = \rho_A^2 (\ln A_0 - \bar{A}) + \rho_A \varepsilon_{A,1} + \varepsilon_{A,2}$$

Finally, substituting equation (10) into equation (8) gives us:

$$(11) \ln A_2 = \bar{A} + 2g + \rho_A^2 (\ln A_0 - \bar{A}) + \rho_A \varepsilon_{A,1} + \varepsilon_{A,2}$$

Now, in period 3 -- using equations (1) and (2) -- we have:

$$(12) \ln A_3 = \bar{A} + 3g + \tilde{A}_3 \quad \text{and} \quad (13) \tilde{A}_3 = \rho_A \tilde{A}_2 + \varepsilon_{A,3}$$

Substituting equation (10) into equation (13) yields:

$$(14) \tilde{A}_3 = \rho_A [\rho_A^2 (\ln A_0 - \bar{A}) + \rho_A \varepsilon_{A,1} + \varepsilon_{A,2}] + \varepsilon_{A,3} = \rho_A^3 (\ln A_0 - \bar{A}) + \rho_A^2 \varepsilon_{A,1} + \rho_A \varepsilon_{A,2} + \varepsilon_{A,3}$$

Finally, substituting equation (14) into equation (12) gives us:

$$(15) \ln A_3 = \bar{A} + 3g + \rho_A^3 (\ln A_0 - \bar{A}) + \rho_A^2 \varepsilon_{A,1} + \rho_A \varepsilon_{A,2} + \varepsilon_{A,3}$$

b) Using equation (7) to find the expected value of $\ln A_1$ yields:

$$E(\ln A_1) = \bar{A} + g + \rho_A (\ln A_0 - \bar{A})$$

since $E(\varepsilon_{A,1}) = 0$.

Using equation (11) to find the expected value of $\ln A_2$ yields:

$$E(\ln A_2) = \bar{A} + 2g + \rho_A^2 (\ln A_0 - \bar{A})$$

since $E(\rho_A \varepsilon_{A,1}) = \rho_A E(\varepsilon_{A,1}) = 0$, $E(\varepsilon_{A,2}) = 0$.

Using equation (15) to find the expected value of $\ln A_3$ yields:

$$E(\ln A_3) = \bar{A} + 3g + \rho_A^3 (\ln A_0 - \bar{A})$$

since $E(\rho_A^2 \varepsilon_{A,1}) = \rho_A^2 E(\varepsilon_{A,1}) = 0$, $E(\rho_A \varepsilon_{A,2}) = \rho_A E(\varepsilon_{A,2}) = 0$, $E(\varepsilon_{A,3}) = 0$.

also means a drop in $i - \pi^e$, since π^e is assumed fixed – then increases planned expenditure. All else equal, the bigger is $E_{i,\pi}$ (in absolute value), the more that planned expenditure will rise – and thus the more that output will end up needing to rise to re-establish the equilibrium condition that $Y = E$.

Problem 5.2

a) The equilibrium condition in the money market is $M/P = L(i, Y)$. Now suppose that the central bank has a target interest rate of $i = \bar{i}$. At a given level of Y , demand for real money balances at the target interest rate would be $L(\bar{i}, Y)$. To ensure money market equilibrium, the central bank will simply have to adjust the nominal money supply, M , to ensure that $M/P = L(\bar{i}, Y)$. Thus the "LM curve" – the set of all combinations of i and Y that cause real money demand and supply to be equal – will be horizontal at the central bank's target level of the interest rate, $\bar{i}$.

b) The AD curve will now be vertical. First of all, why is AD downward sloping when the central bank targets M rather than i ? At a lower P , M/P is higher and thus the LM curve shifts down to the right. Thus the level of Y that clears the money market and equates planned and actual expenditure is now higher. The AD curve is all the combinations of P and Y that clear the money market and equate planned and actual expenditure. Hence lower levels of P are associated with higher levels of Y along the AD curve – AD is downward sloping.

Now suppose the central bank is targeting the interest rate at $i = \bar{i}$. At a lower P , the real money supply would be higher if the central bank did not change M . But the central bank will simply lower M so that at the given combination of income and $\bar{i}$, $M/P = L(\bar{i}, Y)$ again. Thus a lower P will not require a higher level of Y to clear the money market and equate planned and actual expenditure. That is, regardless of the price level, there is one unique level of Y that clears the money market and makes planned and actual expenditure equal – the level of Y where the horizontal "LM curve" and IS intersect – and thus the AD curve is vertical.

Problem 5.3

a) i) How does an equal increase in G and T – a "balanced budget" increase in government purchases – affect the position of the IS curve? We need to determine the effect on Y for a given i to examine the extent of the horizontal shift of the IS curve.

Differentiate the IS curve – $Y = C(Y - T) + I(i - \pi^e) + G$ – with respect to G :

$$\frac{dY}{dG} = \frac{\partial C}{\partial (Y - T)} \left[\frac{dY}{dG} - \frac{dT}{dG} \right] + \frac{\partial I}{\partial (i - \pi^e)} \frac{d(i - \pi^e)}{dG} + 1$$

Although continuing to use total derivative notation, keep in mind that we are now holding π^e and i constant. In addition, we are assuming that $dT = dG$ and so:

$$\frac{dY}{dG} = C_{Y-T} \left[\frac{dY}{dG} - 1 \right] + 1 \Rightarrow \frac{dY}{dG} (1 - C_{Y-T}) = 1 - C_{Y-T}$$

or simply:

$$\left. \frac{dY}{dG} \right|_{i \text{ fixed}} = 1$$

This can be interpreted as meaning the change in Y for a given i is simply equal to the change in government purchases. Thus a change in government purchases – when accompanied by an equal change in taxes – causes the IS curve to shift by a horizontal amount equal to that change in government purchases.

ii) Now we have to allow for a variable interest rate. We want to know the extent of the horizontal shift of the AD curve. That is, we want to know the effect on Y for a given P , but allowing for the effects of a variable interest rate.

Differentiate the IS curve equation -- $Y = C(Y - T) + I(i - \pi^e) + G$ -- with respect to G :

$$\frac{dY}{dG} = \frac{\partial C}{\partial(Y-T)} \left[\frac{dY}{dG} - \frac{dT}{dG} \right] + \frac{\partial I}{\partial(i - \pi^e)} \left[\frac{di}{dG} - \frac{d\pi^e}{dG} \right] + 1$$

We are holding π^e (and P) constant, while assuming $dT = dG$ so we have:

$$(1) \quad \frac{dY}{dG} = C_{Y-T} \frac{dY}{dG} - C_{Y-T} + I_{i-\pi^e} \frac{di}{dG} + 1$$

Differentiate the LM equation -- $M/P = L(i, Y)$ -- with respect to G , holding M and P constant:

$$0 = L_i \frac{di}{dG} + L_Y \frac{dY}{dG}$$

and rearranging to solve for di/dG yields:

$$(2) \quad \frac{di}{dG} = -\frac{L_Y}{L_i} \frac{dY}{dG}$$

Substituting equation (2) into equation (1) yields:

$$\frac{dY}{dG} = C_{Y-T} \frac{dY}{dG} - C_{Y-T} - \frac{I_{i-\pi^e} L_Y}{L_i} \frac{dY}{dG} + 1$$

Solving for dY/dG , we have:

$$(3) \quad \left. \frac{dY}{dG} \right|_{P \text{ fixed}} = \frac{1 - C_{Y-T}}{1 - C_{Y-T} + (I_{i-\pi^e} L_Y / L_i)} < 1$$

The horizontal shift of the AD curve is thus $\left[(1 - C_{Y-T}) / (1 - C_{Y-T} + I_{i-\pi^e} L_Y / L_i) \right] \cdot dG < dG$. The AD curve shifts less than the IS curve. This is because the shift of the AD takes into account the rising interest rate caused by the upward sloping LM.

b) i) We are now assuming that tax revenues are a function of income, $T = T(Y)$. In addition, $T'(Y) > 0$ so that tax revenues rise when income rises. To find the slope of the IS curve, differentiate

$Y = E(Y, i - \pi^e, G, T(Y))$ with respect to i , holding everything else constant:

$$\frac{dY}{di} = E_Y \frac{dY}{di} + E_{i-\pi^e} + E_T T'(Y) \frac{dY}{di} \Rightarrow \frac{dY}{di} [1 - E_Y - E_T T'(Y)] = E_{i-\pi^e}$$

or simply:

$$\frac{dY}{di} = \frac{E_{i-\pi^e}}{1 - E_Y - E_T T'(Y)}$$

Inverting the above expression yields the slope of the IS curve:

$$(1) \quad \left. \frac{di}{dY} \right|_{IS} = \frac{1 - E_Y - E_T T'(Y)}{E_{i-\pi^e}} < 0$$

To see how an increase in $T'(Y)$ affects the slope, take the derivative of the expression for the slope with respect to $T'(Y)$:

$$(2) \frac{\partial (di/dY|_{IS})}{\partial T'(Y)} = -\frac{E_T}{E_{i-\pi^e}} < 0 \quad \text{since } E_T, E_{i-\pi^e} < 0$$

Thus an increase in $T'(Y)$ causes the slope of the IS curve to become even more negative. That is, it causes the IS curve to become steeper.

ii) We need to determine how an increase in $T'(Y)$ affects the impact on output -- for a given price -- of a change in G . First, we need the effect of a change in G on Y , for a given P . Differentiate the IS equation -- $Y = E(Y, i - \pi^e, G, T(Y))$ -- with respect to G , holding π^e (and P) constant:

$$(3) \frac{dY}{dG} = E_Y \frac{dY}{dG} + E_{i-\pi^e} \frac{di}{dG} + 1 + E_T T'(Y) \frac{dY}{dG}$$

Differentiate the LM equation -- $M/P = L(i, Y)$ -- with respect to G , holding P and M constant:

$$0 = L_i \frac{di}{dG} + L_Y \frac{dY}{dG} \Rightarrow \frac{di}{dG} = -\frac{L_Y}{L_i} \frac{dY}{dG} \quad (4)$$

Substitute equation (4) into equation (3):

$$\frac{dY}{dG} = E_Y \frac{dY}{dG} - \frac{E_{i-\pi^e} L_Y}{L_i} \frac{dY}{dG} + E_G + E_T T'(Y) \frac{dY}{dG}$$

Solving for dY/dG yields:

$$(5) \left. \frac{dY}{dG} \right|_{P \text{ fixed}} = \frac{E_G}{1 - E_Y + (E_{i-\pi^e} L_Y / L_i) - E_T T'(Y)} > 0$$

Now to see how the impact of a change in government purchases on output -- for a given price level -- is affected by a rise in $T'(Y)$:

$$(6) \frac{\partial (dY/dG|_{P \text{ fixed}})}{\partial T'(Y)} = \frac{-E_G}{[1 - E_Y + (E_{i-\pi^e} L_Y / L_i) - E_T T'(Y)]^2} \cdot (-E_T) < 0 \quad \text{since } E_G > 0, E_T < 0$$

When $T'(Y)$ is higher, the increase in Y -- for a given P -- due to an increase in G is smaller. Thus we have shown that the horizontal shift of the AD curve due to a change in government purchases will be smaller, the larger is $T'(Y)$.

Now we need to determine how an increase in $T'(Y)$ affects the impact on output -- for a given price -- of a change in M . First, we need the effect of a change in M on Y , for a given P . Differentiate the IS equation -- $Y = E(Y, i - \pi^e, G, T(Y))$ -- with respect to M , holding G , π^e (and P) constant:

$$(7) \frac{dY}{dM} = E_Y \frac{dY}{dM} + E_{i-\pi^e} \frac{di}{dM} + E_T T'(Y) \frac{dY}{dM}$$

Differentiate the LM equation -- $M/P = L(i, Y)$ -- with respect to M , holding P constant:

$$\frac{1}{P} = L_i \frac{di}{dM} + L_Y \frac{dY}{dM} \Rightarrow \frac{di}{dM} = \frac{1}{PL_i} - \frac{L_Y}{L_i} \frac{dY}{dM} \quad (8)$$

Substitute equation (8) into equation (7):

$$\frac{dY}{dM} = E_Y \frac{dY}{dM} + \frac{E_{i-\pi^e}}{PL_i} - \frac{E_{i-\pi^e} L_Y}{L_i} \frac{dY}{dM} + E_T T'(Y) \frac{dY}{dM}$$

Solving for dY/dM yields:

$$(9) \left. \frac{dY}{dM} \right|_{P \text{ fixed}} = \frac{E_{i-\pi^e}}{PL_i \left[1 - E_Y + \left(E_{i-\pi^e} L_Y / L_i \right) - E_T T'(Y) \right]} > 0$$

To see how the impact of a change in M on Y — for a given P — is affected by a rise in $T'(Y)$, look at the derivative:

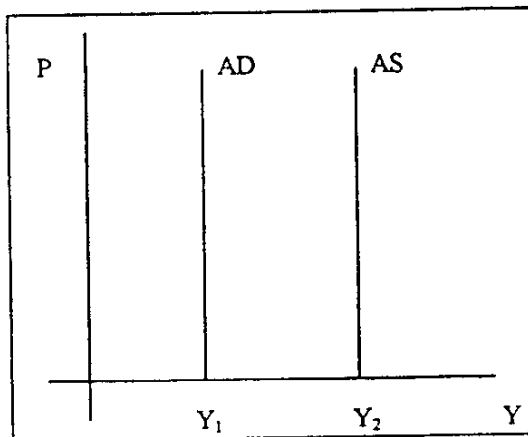
$$(10) \frac{\partial \left(\left. \frac{dY}{dM} \right|_{P \text{ fixed}} \right)}{\partial T'(Y)} = \frac{-E_{i-\pi^e} (-PL_i E_T)}{(PL_i)^2 \left[1 - E_Y + \left(E_{i-\pi^e} L_Y / L_i \right) - E_T T'(Y) \right]^2} < 0$$

This derivative is negative since $E_{i-\pi^e}$, L_i and E_T are all negative. When $T'(Y)$ is higher, the increase in Y — for a given P — from an increase in M is smaller. Thus the horizontal shift of the AD curve due to a change in the nominal quantity of money will be smaller when $T'(Y)$ is higher.

Problem 5.4

a) In a liquidity trap, the LM curve is horizontal at the low nominal interest rate that prevails. Intuitively, at a given interest rate, people are willing to hold any quantity of real money balances. Thus at higher levels of income (Y) people are willing to change their real money balances without a change in the interest rate. Thus higher levels of Y no longer require higher i 's to clear the money market. That is, LM is horizontal rather than upward sloping. This means that the AD curve will be vertical. Go through the procedure of deriving the AD curve. A decrease in P increases the supply of real money balances. Ordinarily, a lower interest rate is needed to clear the money market for a given level of income and so the LM curve would shift down. As a result, i falls and Y rises. Thus the level of output at the intersection of the IS and LM curves is ordinarily a decreasing function of the price level. That is, the AD curve is usually downward sloping — lower levels of P require higher levels of Y to clear the money market and make planned expenditure equal actual expenditure.

In the case of a liquidity trap, the decrease in P — which increases the supply of real money balances, M/P — does not shift the horizontal LM curve. At a given level of income, we do not require a lower i to get people to hold the extra money. People are willing to hold any quantity of real money balances at a given level of Y . Thus LM does not shift down. The same level of Y continues to clear the money market and make planned expenditure equal actual expenditure, regardless of the price level. In other words, the level of output at the intersection of the IS and LM curves is no longer a function of the price level — AD is vertical.



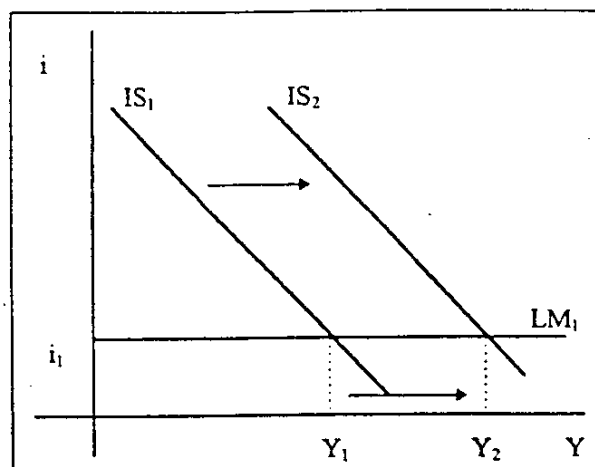
Now, even if prices are fully flexible (so that the AS curve is vertical), aggregate demand can play a role in determining output. Suppose the AD curve lies to the left of the AS curve as is depicted at right. Then output bought and sold in the economy will be at Y_1 , not Y_2 . We have a situation of excess supply that cannot be eliminated by changes in price, since price does not affect AD.

b) We can now write planned expenditure as:

$$(1) E = E(Y, i - \pi^e, G, T, M/P) \quad E_{M/P} > 0$$

The important point is that even if there is a liquidity trap -- so that LM is horizontal -- the AD curve will once again be downward sloping.

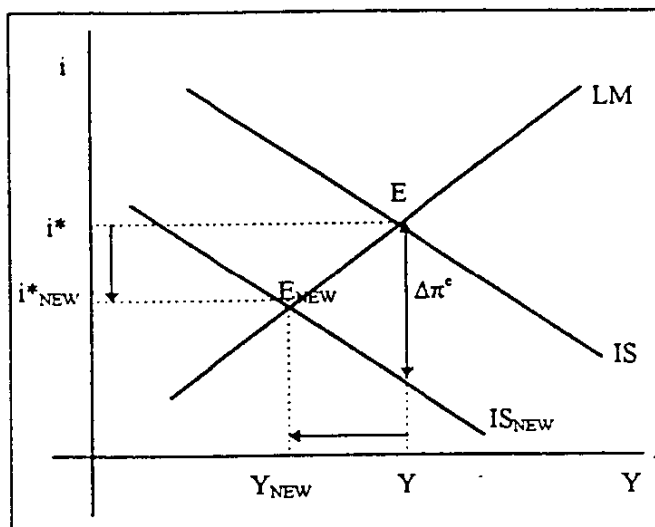
Go through the exercise of deriving the AD curve. Suppose IS_1 and LM_1 correspond to some price level P_1 . Now consider a lower price level P_2 . As explained in part (a), the horizontal LM will not shift. But the IS curve will. The fall in P raises M/P which is a component of wealth. With $E_{M/P} > 0$, this increase in wealth increases planned expenditure at a given i . In a "Keynesian Cross" diagram, the planned expenditure line would shift up. Thus the level of Y required to equate planned and actual expenditure is now higher at a given i . That is, the IS curve shifts to the right to IS_2 .



This IS/LM diagram then tells us that at lower P 's, the level of Y that clears the money market and equates planned and actual expenditure, is now higher. Thus the AD curve -- which is all the combinations of P and Y such that the money market clears and planned and actual expenditure are equal -- is once again downward sloping. In this case -- a liquidity trap combined with real money holdings affecting planned expenditure -- a vertical AS curve once again means that AD is irrelevant to output. Essentially, we have re-introduced a way in which prices affect AD. Thus any situation of excess supply can be eliminated by a falling price level.

Problem 5.5

A fall in expected inflation shifts the IS curve to the left to IS_{NEW} . At a given nominal interest rate, i , the real interest rate -- $i - \pi^e$ -- is now higher and thus planned expenditure is lower. In a "Keynesian Cross" diagram, the planned expenditure line would shift down. Thus the level of Y required to equate planned and actual expenditure is now lower at a given i . That is, IS shifts left.



From the graph at right, we can see that the fall in π^e causes both output and the nominal interest rate to fall. What about the real interest rate, $i - \pi^e$? It rises -- that is, i falls less than π^e does. Graphically, think of the IS curve as shifting down. What is the magnitude of the downward shift in IS? It must be the change in expected inflation, $\Delta\pi^e$. Intuitively, the same level of Y would continue to equate planned and actual expenditure if i were to fall by the same amount as π^e so that $i - \pi^e$ (and thus planned expenditure) remained unchanged. Thus on the new IS curve, a given level of Y is now associated with a nominal interest rate that is lower by the amount $\Delta\pi^e$ -- IS shifts down by $\Delta\pi^e$. We can see from the graph that because of the

upward sloping LM curve, i does not fall by the full extent of the downward shift in IS. Thus i falls by less than π^e does and so the real interest rate, $i - \pi^e$, rises.

More formally, the IS and LM equations are given by:

$$(1) Y = E(Y, i - \pi^e, G, T) \quad \text{and} \quad (2) M/P = L(i, Y)$$

Differentiate the IS equation with respect to π^e , holding G and T constant:

$$\frac{dY}{d\pi^e} = E_Y \frac{dY}{d\pi^e} + E_{i-\pi^e} \left[\frac{di}{d\pi^e} - 1 \right]$$

Rearranging to solve for $di/d\pi^e$ gives us:

$$E_{i-\pi^e} \frac{di}{d\pi^e} = (1 - E_Y) \cdot \frac{dY}{d\pi^e} \Rightarrow \frac{di}{d\pi^e} = \frac{(1 - E_Y)}{E_{i-\pi^e}} \cdot \frac{dY}{d\pi^e} + 1 \quad (3)$$

Differentiate the LM equation with respect to π^e , holding M and P constant:

$$0 = L_i \frac{di}{d\pi^e} + L_Y \frac{dY}{d\pi^e} \Rightarrow \frac{dY}{d\pi^e} = \frac{-L_i}{L_Y} \cdot \frac{di}{d\pi^e} \quad (4)$$

Substitute equation (4) into equation (3):

$$\frac{di}{d\pi^e} = \frac{(1 - E_Y)}{E_{i-\pi^e}} \cdot \frac{-L_i}{L_Y} \cdot \frac{di}{d\pi^e} + 1 \Rightarrow \frac{di}{d\pi^e} \left[\frac{E_{i-\pi^e} \cdot L_Y + (1 - E_Y) L_i}{E_{i-\pi^e} \cdot L_Y} \right] = 1$$

and thus:

$$(5) \frac{di}{d\pi^e} = \frac{E_{i-\pi^e} \cdot L_Y}{E_{i-\pi^e} \cdot L_Y + (1 - E_Y) L_i} > 0$$

Note that $|di/d\pi^e| < 1$ which confirms the graphical and intuitive analysis above. The nominal interest rate falls less than expected inflation does and so the real interest rate, $i - \pi^e$, rises.

Substituting equation (5) into equation (4) gives us an expression for the change in output due to the change in expected inflation:

$$(6) \frac{dY}{d\pi^e} = \frac{-L_i}{L_Y} \cdot \frac{E_{i-\pi^e} \cdot L_Y}{E_{i-\pi^e} \cdot L_Y + (1 - E_Y) L_i} = \frac{-L_i E_{i-\pi^e}}{E_{i-\pi^e} \cdot L_Y + (1 - E_Y) L_i} > 0$$

Since $dY/d\pi^e > 0$, a fall in expected inflation causes output to fall. This confirms the graphical analysis above.

Problem 5.6

a) Substituting the consumption function -- $C_t = a + bY_{t-1}$ -- and the assumption about investment -- $I_t = K_t^* - cY_{t-2}$ -- into the equation for output -- $Y_t = C_t + I_t + G_t$ -- yields:

$$(1) Y_t = a + bY_{t-1} + K_t^* - cY_{t-2} + G_t$$

Substituting for the desired capital stock -- $K_t^* = cY_{t-1}$ -- and for the constant level of government purchases -- $G_t = \bar{G}$ -- yields:

$$(2) Y_t = a + bY_{t-1} + cY_{t-1} - cY_{t-2} + \bar{G}$$

Collecting terms in Y_{t-1} gives us output in period t as a function of Y_{t-1} , Y_{t-2} and the parameters of the model:

$$(3) Y_t = a + (b + c)Y_{t-1} - cY_{t-2} + \bar{G}$$

b) With the assumptions of $b = 0.9$ and $c = 0.5$, output in period t is given by:

$$(4) Y_t = a + 1.4Y_{t-1} - 0.5Y_{t-2} + \bar{G}$$

Throughout the following, the change in a variable represents the change from the path that variable would have taken if G had simply remained constant at $\bar{G}$.

In period t :

$$Y_t = a + 1.4Y_{t-1} - 0.5Y_{t-2} + \bar{G} + 1$$

and thus the change in output from the path it would have taken is given by:

$$\Delta Y_t = +1$$

In period $t + 1$, using the fact that equation (4) will hold in all future periods:

$$\Delta Y_{t+1} = 1.4 \cdot \Delta Y_t - 0.5 \cdot \Delta Y_{t-1} = 1.4(+1) - 0.5(0) = +1.4$$

In period $t + 2$:

$$\Delta Y_{t+2} = 1.4 \cdot \Delta Y_{t+1} - 0.5 \cdot \Delta Y_t = 1.4(+1.4) - 0.5(+1) = +1.46$$

In period $t + 3$:

$$\Delta Y_{t+3} = 1.4 \cdot \Delta Y_{t+2} - 0.5 \cdot \Delta Y_{t+1} = 1.4(+1.46) - 0.5(+1.4) = +1.344$$

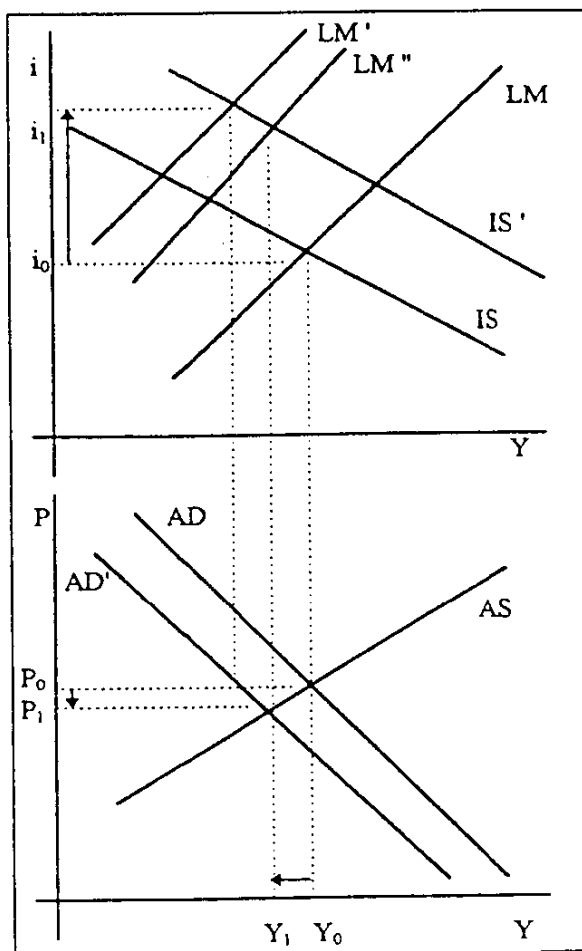
With similar calculations, one can show that $\Delta Y_{t+4} = +1.15$, $\Delta Y_{t+5} = 0.938$ etc. Thus output follows a "hump-shaped" response to the one-time increase in government purchases of 1. The maximum effect is felt two periods after the increase in G and the effect then goes to 0 over time.

Problem 5.7

A decrease in taxes means that at a given i , planned expenditure is higher since $E_T < 0$. Thus at a given i , Y must be higher in order for $Y = E$. Thus the IS curve shifts to the right to IS' . Ordinarily, that would be the whole story. The AD curve would shift to the right and the tax cut would increase the level of output. Here, there is also an effect on the LM curve. A decrease in taxes means that at a given Y , disposable income, $(Y - T)$, is higher. Thus so is real money demand. At a given Y , this requires a higher i than before to have real money demand equal real money supply. So the LM curve shifts up to LM' .

There is an ambiguity. The LM curve could shift up so much that — as in the case depicted in the diagram at right — the intersection of IS' and LM' occurs to the left of the original Y_0 . That is, Y for a given P falls and so the AD curve would actually shift to the left in this case. Thus the tax cut could end up reducing output. [Note that the fall in P increases the real money supply and thus shifts LM down to LM'' to complete the story].

Determining whether output rises or falls is equivalent to determining whether the AD curve shifts to the left or to the right. So we can see what happens to Y , for a given P , and that will answer the question of whether output rises or falls in the end.



Differentiate the LM relationship — $M/P = L(i, Y - T)$ — with respect to T , holding M and P constant:

$$0 = L_i \frac{di}{dT} + L_{Y-T} \left(\frac{dY}{dT} - 1 \right)$$

Rearranging to solve for di/dT yields:

$$(1) \frac{di}{dT} = \frac{L_{Y-T}}{L_i} - \frac{L_{Y-T}}{L_i} \frac{dY}{dT}$$

Differentiate the IS relationship $-Y = E(Y, i - \pi^e, G, T)$ -- with respect to T , holding π^e , G (and P) constant:

$$(2) \frac{dY}{dT} = E_Y \frac{dY}{dT} + E_{i-\pi^e} \frac{di}{dT} + E_T$$

Substitute equation (1) into equation (2):

$$\frac{dY}{dT} = E_Y \frac{dY}{dT} + E_{i-\pi^e} \left(\frac{L_{Y-T}}{L_i} - \frac{L_{Y-T}}{L_i} \frac{dY}{dT} \right) + E_T$$

Solving for dY/dT yields:

$$(3) \frac{dY}{dT} \Big|_{P \text{ fixed}} = \frac{(E_{i-\pi^e} L_{Y-T} / L_i) + E_T}{1 - E_Y + (E_{i-\pi^e} L_{Y-T} / L_i)}$$

The denominator of equation (3) is positive since $E_{i-\pi^e} < 0$, $L_{Y-T} > 0$, $L_i < 0$ and $E_Y < 1$. The sign of the numerator is indeterminate. The first term is negative and is basically capturing the money market effects of the cut in T . The second term is positive and is basically capturing the effects in the goods market of the cut in T . If the former effect is stronger then dY/dT for a given P will be positive, meaning a cut in T reduces Y for a given P . This is equivalent to saying that the AD curve shifts to the left.

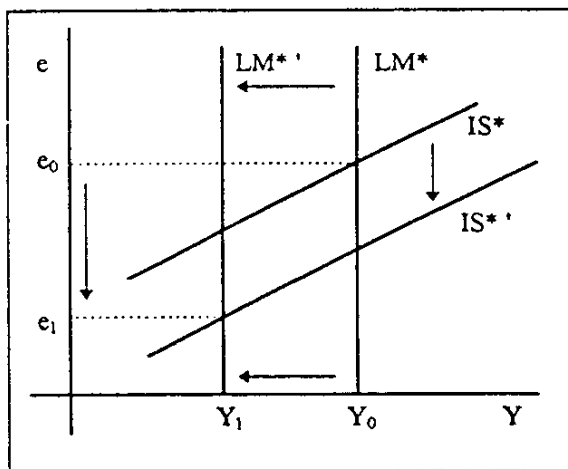
b) We will again restrict ourselves to looking at the effects on Y for a given P .

Due to the tax cut, at a given e , planned expenditure is higher -- $E_T < 0$ -- and therefore we need a higher Y at a given e to satisfy the equilibrium condition that $Y = E$. Thus IS^* shifts to the right to $IS^{*'}$.

Ordinarily that would be the whole story. There would be an appreciation causing a reduction in planned expenditure by an amount equal to the rise in planned expenditure caused by the tax cut.

Thus on net, planned expenditure and thus Y would not change. But here, at the original Y_0 , $Y_0 - T$ is now higher than before due to the tax cut. Therefore, at the original Y_0 , real money demand would be higher, throwing the money market out of equilibrium. In order to clear the money market -- since i^* cannot change -- we need a lower Y so that $Y - T$ and thus real money demand is unchanged and still equal to the given M/P . Thus, LM^* shifts to the left to $LM^{*'}$. The end result is that output falls from Y_0 to Y_1 . A tax cut actually reduces the level of output for a given P . Formally, differentiate the LM^* relationship -- $M/P = L(i^*, Y - T)$ -- with respect to T , holding M , P and i^* constant:

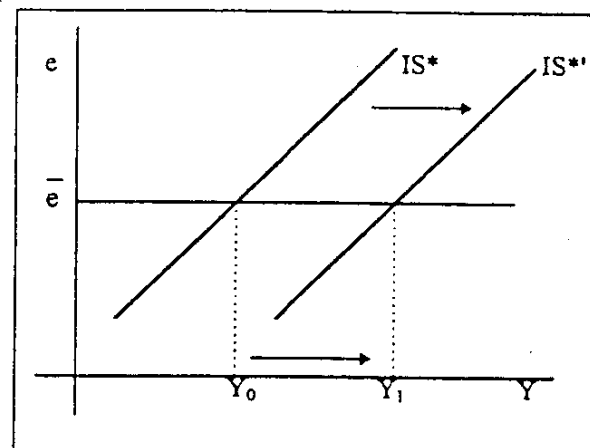
$$0 = L_{Y-T} \left(\frac{dY}{dT} - 1 \right) \Rightarrow \frac{dY}{dT} \Big|_{P \text{ fixed}} = 1$$



Output is determined entirely in the money market in this model. In response to a disturbance — since i^* , M and P are constant — it must be Y that adjusts to ensure equilibrium. As T falls, Y must fall by an equal amount.

c) Again we will look at the effects on Y for a given P . We are assuming the central bank will ensure money market equilibrium. They will do so by meeting money demand with the appropriate money supply to ensure that the nominal exchange rate remains fixed. Thus our alteration to the money demand function does not mean that the nominal exchange rate will be affected by tax changes — it simply means that the central bank will have to act to offset the effects of tax changes on money demand.

After T falls, IS^* shifts to the right for the same reason as described above. Any increase in money demand that is caused by T falling is simply met with an increase in the money supply by the central bank — M is no longer exogenous. We no longer need a drop in Y to maintain money market equilibrium. The end result is that Y rises to Y_1 . We get the "usual" result that a tax cut increases output for a given P .



Problem 5.8

Planned expenditure is given by:

$$(1) E = C(Y - T) + I(i - \pi^*) + G + NX(\epsilon P^*/P)$$

For a floating exchange rate and perfect capital mobility, we have:

$$(2) Y = C(Y - T) + I(i^* - \pi^*) + G + NX(\epsilon P^*/P) \quad IS^* \text{ Curve}$$

$$(3) M/P = L(i^*, Y) \quad LM^* \text{ Curve}$$

For a fixed exchange rate and perfect capital mobility, we have equation (2) for the IS^* curve and the exchange rate equation:

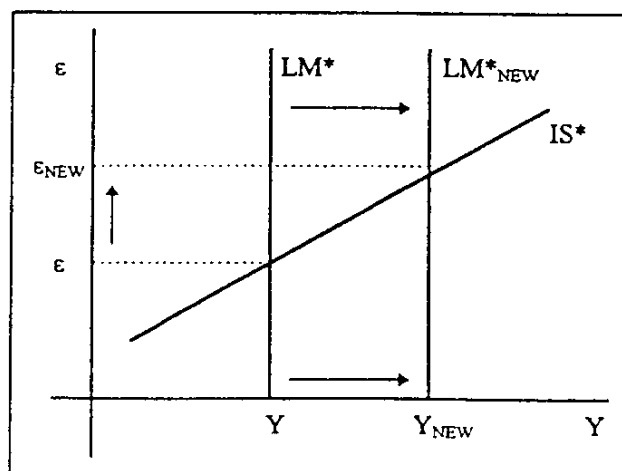
$$(4) \epsilon = \bar{\epsilon}$$

For a floating exchange rate and imperfect capital mobility, we have:

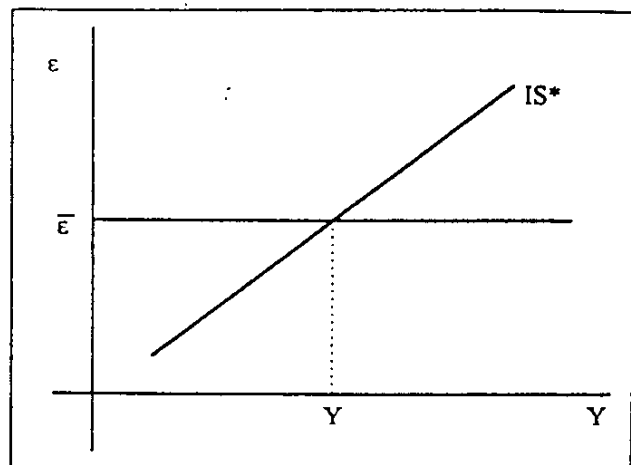
$$(5) Y = C(Y - T) + I(i - \pi^*) + G - CF(i - i^*) \quad IS^{**} \text{ Curve}$$

$$(6) M/P = L(i, Y) \quad LM \text{ Curve}$$

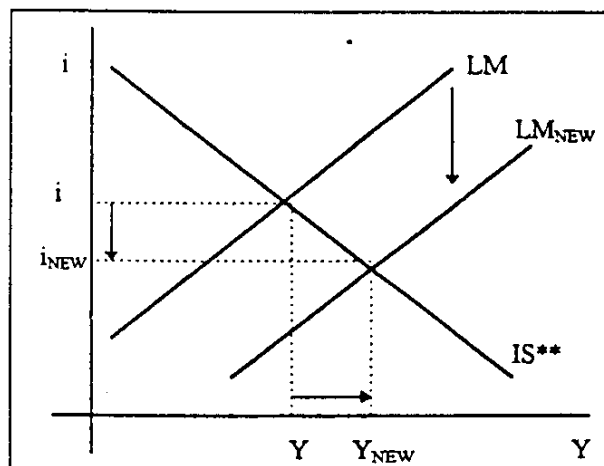
a) i) From equation (3), with M/P unchanged, $L(i^*, Y)$ must be unchanged as well for the money market to remain in equilibrium. Since i^* does not change and because $L_Y > 0$, Y must rise so that money demand returns to its original value at that given i^* . Thus the LM^* curve shifts to the right to LM_{NEW}^* . The IS^* curve is unaffected — money demand does not appear in equation (2). Thus for a given P , income rises to Y_{NEW} and ϵ rises to ϵ_{NEW} (the domestic currency depreciates). Finally, since $NX_{\epsilon P^*/P} > 0$, the rise in ϵ increases net exports for a given P .



ii) Again, the drop in the demand for money does not affect the IS^* curve. In addition, the nominal exchange rate remains pegged at $\bar{\epsilon}$. The central bank simply reduces the money supply to match the decrease in money demand. Thus for a given P , income, the exchange rate and net exports are all unaffected. With a fixed exchange rate, disturbances in the money market have no impact on Y for a given P .



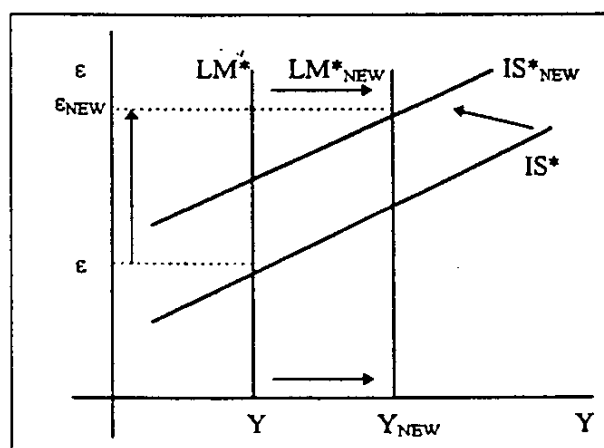
iii) From equation (5), the IS^{**} curve is unaffected. The LM curve shifts down. At a given level of income, the interest rate must be lower in order to drive money demand back up and keep the money market in equilibrium. From the diagram, income for a given P rises. Since the domestic interest rate is now lower and since $CF'(i - i^*) > 0$, capital flows are lower. Thus NX must be higher in order for the balance of payments to be 0. Since $NX_{\epsilon P/P} > 0$, ϵ must be higher. That is, the domestic currency must have depreciated.



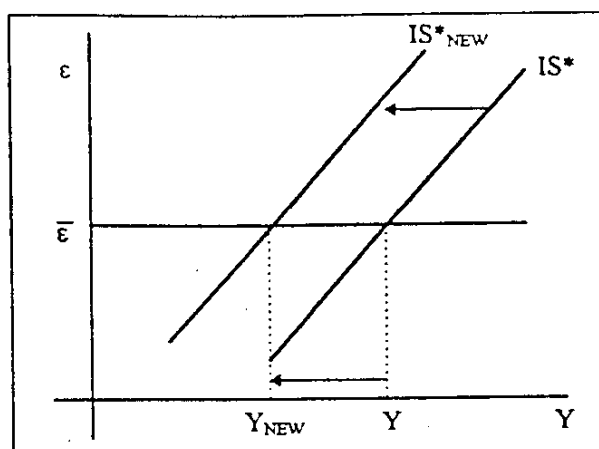
b) The foreign interest rate rises.

i) Since $L_{i^*} < 0$, the rise in i^* tends to reduce real money demand. To offset this and keep the money market in equilibrium, Y must be higher. Since $L_Y > 0$, a higher Y will keep real money demand unchanged and equal to the unchanged real money supply. Thus the LM^* curve shifts to the right.

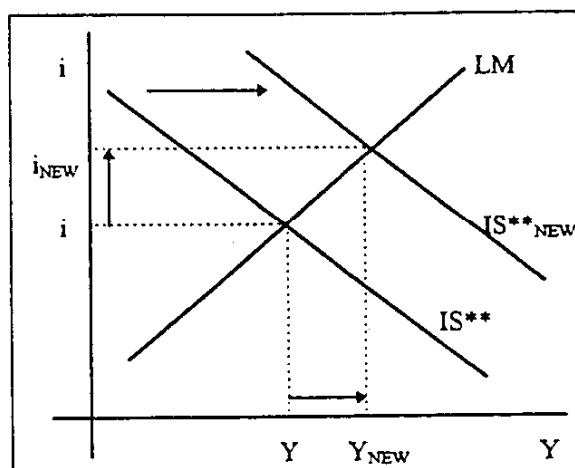
From equation (2), the rise in i^* reduces planned expenditure at a given ϵ since $I_{i-\pi^*} < 0$. This reduces the level of Y that equates planned and actual expenditure at a given ϵ . In other words, the IS^* curve shifts to the left. Thus income for a given P rises. The exchange rate rises — the domestic currency depreciates. Finally, since $NX_{\epsilon P/P} > 0$, net exports rise due to the depreciation.



ii) Again, the IS^* curve shifts to the left due to the rise in the foreign interest rate. The nominal exchange rate remains pegged at $\bar{\epsilon}$. The central bank is offsetting the effects on money demand with changes in the nominal money supply. Thus for a given P , income falls. Since the exchange rate does not change, neither do net exports.



iii) From equation (6), the LM curve is unaffected by the rise in i^* . At a given i , the rise in i^* reduces capital flows and thus – in order for the balance of payments to equal 0 – must increase net exports. Thus at a given i , planned expenditure is now higher. Thus at a given i , the level of Y that equates planned and actual expenditure is higher. The IS^{**} curve shifts to the right. At a given P , the level of income rises.



One way to see the effects on NX and ϵ for a given P is the following. Differentiate both sides of equation (6) – the LM curve – with respect to i^* , holding M and P constant:

$$0 = L_i \cdot \frac{di}{di^*} + L_Y \cdot \frac{dY}{di^*}$$

Solving for dY/di^* yields:

$$(7) \quad \frac{dY}{di^*} = -\frac{L_i}{L_Y} \cdot \frac{di}{di^*}$$

Differentiate both sides of equation (5) – the IS^{**} curve – with respect to i^* , holding T , π^* and G constant:

$$(8) \quad \frac{dY}{di^*} = C_{Y-T} \cdot \frac{dY}{di^*} + I_{i-\pi^*} \cdot \frac{di}{di^*} - CF'(i-i^*) \cdot \left[\frac{di}{di^*} - 1 \right]$$

Substitute equation (7) into equation (8):

$$-\frac{L_i}{L_Y} \cdot \frac{di}{di^*} = -C_{Y-T} \cdot \frac{L_i}{L_Y} \cdot \frac{di}{di^*} + I_{i-\pi^*} \cdot \frac{di}{di^*} - CF'(i-i^*) \cdot \frac{di}{di^*} + CF'(i-i^*)$$

Collecting the terms in di/di^* gives us:

$$\frac{di}{di^*} \cdot \left\{ -[1 - C_{Y-T}] \cdot (L_i/L_Y) - I_{i-\pi^*} + CF'(i-i^*) \right\} = CF'(i-i^*)$$

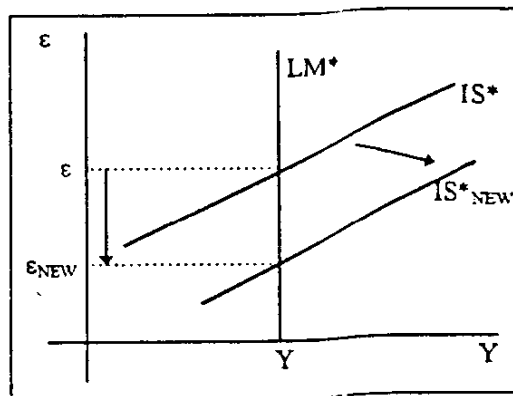
and thus the change in the domestic interest rate due to a change in the foreign interest rate is given by:

$$(9) \quad \frac{di}{di^*} = \frac{CF'(i-i^*)}{CF'(i-i^*) - I_{i-\pi^*} - [1 - C_{Y-T}] \cdot (L_i/L_Y)}$$

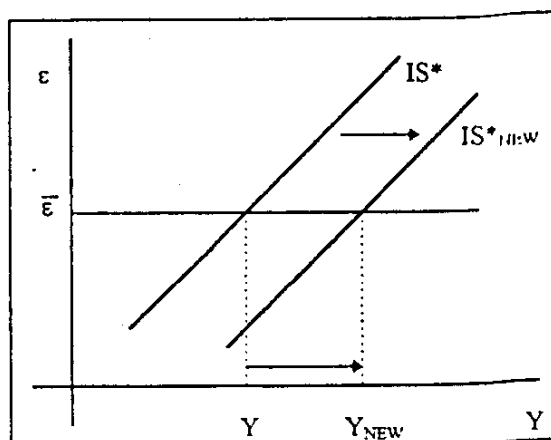
Note that $CF'(i^*) > 0$, $I_{i-\pi^e} < 0$, $[1 - C_{Y-T}] > 0$, $L_i < 0$ and $L_Y > 0$. Thus $di/di^* < 1$. So i rises less than i^* -- that is, $i - i^*$ falls, and thus capital flows decrease. This means that NX must increase at a given P to ensure a balance of payments equal to 0. Finally, since $NX_{\epsilon P^*P} > 0$, if NX rises it must be the case that ϵ rises at a given P . That is, the domestic currency depreciates.

c) The country adopts protectionist policies so that net exports at a given real exchange rate are higher than before.

i) From equation (3), the LM^* curve is unaffected. Since net exports are higher at a given ϵ , planned expenditure is higher at a given ϵ . Thus the level of Y that equates planned and actual expenditure at a given ϵ is higher. The IS^* curve shifts to the right. For a given P , the level of income is unchanged -- it is determined entirely in the money market here. All that happens is a drop in ϵ -- an appreciation of the domestic currency -- which keeps NX unchanged at a given P . Protectionist policies do not improve the trade balance in this model.



ii) Again, the IS^* curve shifts to the right. The nominal exchange rate remains pegged at $\bar{\epsilon}$. The central bank adjusts the nominal money stock to ensure that the exchange rate does not change. For a given P , income rises.



Even though the exchange rate is unchanged, net exports at a given P wind up higher in the end due to the protectionist policies. Thus unlike with a floating exchange rate, protectionist policies do work with a fixed exchange rate regime.

iii) The LM curve is unaffected by the protectionist policies. In addition, the IS^{**} curve is unaffected -- see equation (5), where NX does not appear. Since $CF(i - i^*)$ is not affected by this policy, NX cannot change in the end either. Thus income for a given P does not change, nor does net exports. What must happen is that the domestic currency appreciates -- ϵ falls -- which offsets the effect of the protectionist policies on net exports. This is the same result obtained with perfect capital mobility and flexible exchange rates.

Problem 5.9

a) With this foreign exchange market intervention, the balance of payments equation is:

$$(1) \quad CF(i - i^*) + NX(Y, i - \pi^e, G, T, \epsilon P^*/P) = a \quad \text{where } a > 0$$

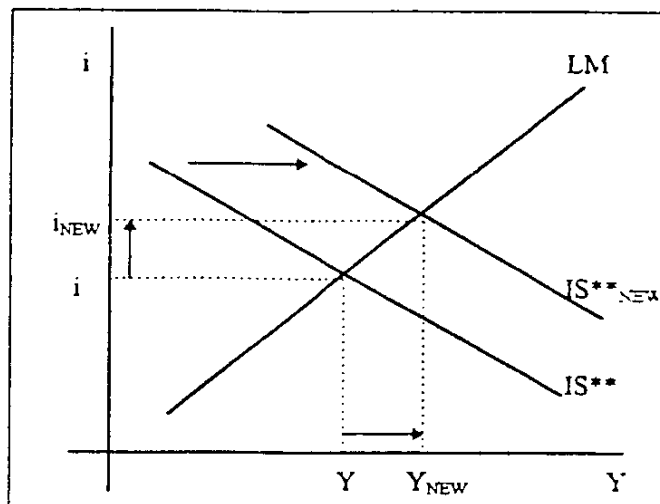
Thus net exports are given by:

$$(2) \quad NX(Y, i - \pi^e, G, T, \epsilon P^*/P) = a - CF(i - i^*)$$

Substituting this expression for net exports into equation (5.22) in the text -- $Y = E^D(Y, i - \pi^e, G, T) + NX(Y, i - \pi^e, G, T, \epsilon P^*/P)$ -- yields the IS^{**} curve:

$$(3) \quad Y = E^D(Y, i - \pi^e, G, T) + a - CF(i - i^*)$$

Compared to the situation of $a = 0$, this raises the level of Y for a given interest rate and thus shifts the IS^{**} curve to the right. This tells us that Y for a given P rises and hence the AD curve shifts to the right. As long as the AS curve is upward sloping, both output and the price level rise in the end. The intervention -- where the central bank sells domestic currency -- causes the domestic currency to depreciate. That is, ϵ rises.



b) If capital is perfectly mobile, sterilized intervention has no effects.

Diagrammatically, the IS^{**} curve is horizontal under perfect capital mobility. Thus a "shift to the right" of the IS^{**} curve does not actually affect its position.

Problem 5.10

a) From the equation describing the behavior of inflation, $\dot{p} = \theta y$, when all prices have adjusted ($\dot{p} = 0$), y must equal 0. Substituting $\dot{p} = 0$, $y = 0$ and $i = \epsilon$ into the IS equation -- $y = b(\epsilon - p) - a(i - \dot{p})$ -- yields:

$$(1) 0 = b(\epsilon - p) - a\epsilon$$

and thus the change in the log exchange rate is:

$$(2) \dot{\epsilon} = (b/a)(\epsilon - p)$$

Since p is constant, if ϵ were greater than p , $\dot{\epsilon} > 0$ and so ϵ would rise without bound. If ϵ were less than p , $\dot{\epsilon} < 0$ and so ϵ would continually fall. Thus when all prices have adjusted, we must have $\epsilon = p$ for ϵ to remain constant. Substituting $y = 0$ and $i = \dot{\epsilon} = 0$ into the LM equation -- $m - p = hy - ki$ -- yields:

$$(3) m = p$$

Thus once prices have fully adjusted, $y = i = 0$ and $m = p = \epsilon$.

b) If ϵ is to jump to exactly m and then remain constant, we need an equilibrium where $\epsilon = m$ and $\dot{\epsilon} = 0$. Since $i = \dot{\epsilon}$, we must have i remain equal to its fully adjusted level of 0. Thus with i constant, y must adjust to ensure money market equilibrium. Substituting $\dot{p} = \theta y$, $\epsilon = m$ and $i = 0$ into the IS equation yields:

$$(4) y = b(m - p) + a\theta y$$

Solving equation (4) for y yields:

$$(5) y = [b/(1 - a\theta)](m - p)$$

Substituting the assumption of $i = 0$ into the LM equation gives us:

$$(6) m - p = hy$$

Substituting equation (6) into equation (5) yields:

$$(7) y = [b/(1 - a\theta)]hy$$

Thus we need the parameters to satisfy:

$$(8) 1 - a\theta = bh$$

or equivalently:

$$(9) a\theta + bh = 1$$

Problem 5.11

a) Taking the derivative of both sides of equation (5.12) – $Y = E(Y, i - \pi^e, G, T, \varepsilon P^*/P)$ – with respect to Y , holding π^e , G , T , P^* and P constant yields:

$$(1) \quad 1 = E_Y + E_{i-\pi^e} \frac{di}{dY} + E_{\varepsilon P^*/P} \frac{d\varepsilon}{dY}$$

Taking the derivative of both sides of the balance of payments equation (5.21) – $CF(i - i^*) + NX(Y, i - \pi^e, G, T, \varepsilon P^*/P) = 0$ – with respect to Y , holding i^* , π^e , G , T , P^* and P constant gives us:

$$(2) \quad CF'(i) \frac{di}{dY} + NX_Y + NX_{i-\pi^e} \cdot \frac{di}{dY} + NX_{\varepsilon P^*/P} \cdot \frac{d\varepsilon}{dY} = 0$$

Rearranging equation (1) to solve for $d\varepsilon/dY$ yields:

$$(3) \quad \frac{d\varepsilon}{dY} = \frac{(1 - E_Y)}{E_{\varepsilon P^*/P}} - \frac{E_{i-\pi^e}}{E_{\varepsilon P^*/P}} \cdot \frac{di}{dY}$$

Substituting equation (3) into equation (2) leaves us with:

$$CF'(i) \frac{di}{dY} + NX_Y + NX_{i-\pi^e} \cdot \frac{di}{dY} + \frac{NX_{\varepsilon P^*/P}(1 - E_Y)}{E_{\varepsilon P^*/P}} - \frac{NX_{\varepsilon P^*/P} E_{i-\pi^e}}{E_{\varepsilon P^*/P}} \cdot \frac{di}{dY} = 0$$

Collecting terms yields:

$$\frac{di}{dY} \left[CF'(i) + NX_{i-\pi^e} - \frac{NX_{\varepsilon P^*/P} E_{i-\pi^e}}{E_{\varepsilon P^*/P}} \right] = - \left[NX_Y + \frac{NX_{\varepsilon P^*/P}}{E_{\varepsilon P^*/P}} (1 - E_Y) \right]$$

Thus finally, the slope of the IS^{**} curve is given by:

$$(4) \quad \left. \frac{di}{dY} \right|_{IS^{**}} = \frac{- \left[NX_Y + \frac{NX_{\varepsilon P^*/P}}{E_{\varepsilon P^*/P}} (1 - E_Y) \right]}{CF'(i) + NX_{i-\pi^e} - \frac{NX_{\varepsilon P^*/P} E_{i-\pi^e}}{E_{\varepsilon P^*/P}}} < 0$$

We are told that $NX_Y + (1 - E_Y) > 0$. Since $NX_{\varepsilon P^*/P} / E_{\varepsilon P^*/P} \geq 1$, the term in brackets in the numerator is also positive and thus the numerator itself is negative. It is straightforward to verify that the denominator is positive and thus IS^{**} remains downward sloping.

b) Rearranging equation (2) to solve for $d\varepsilon/dY|_{IS^{**}}$ yields:

$$(5) \quad \left. \frac{d\varepsilon}{dY} \right|_{IS^{**}} = \frac{- \left[CF'(i) + NX_{i-\pi^e} \right] \left. \frac{di}{dY} \right|_{IS^{**}} - NX_Y}{E_{\varepsilon P^*/P}} > 0$$

$[CF'(i) + NX_{i-\pi^e}]$ is positive and $di/dY|_{IS^{**}}$ is negative. Since NX_Y is negative, the numerator is positive.

The denominator is positive. Thus as we move down the IS^{**} curve and Y rises, ε rises as well. That is, as we move down the IS^{**} curve, the domestic currency depreciates.

c) In order to see the effect of an increase in capital mobility, use equation (4) to take the derivative of the slope of the IS^{**} curve with respect to $CF'(i)$:

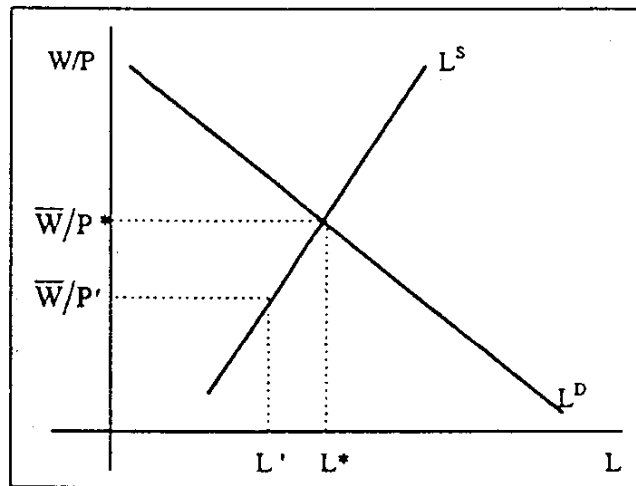
$$\frac{\partial \left(\frac{di}{dY} \right)_{IS^{**}}}{\partial CF'(i)} = \frac{NX_Y + \frac{NX_{\epsilon P^*/P}}{E_{\epsilon P^*/P}} (1 - E_Y)}{\left[CF'(i) + NX_{1-\pi^e} - \frac{NX_{\epsilon P^*/P} E_{1-\pi^e}}{E_{\epsilon P^*/P}} \right]^2} > 0$$

Again, we are told that $NX_Y + (1 - E_Y) > 0$. Since $NX_{\epsilon P^*/P} / E_{\epsilon P^*/P} \geq 1$, the numerator is positive. Thus a rise in $CF'(i)$ – an increase in the degree of capital mobility – causes the slope of the IS^{**} curve to rise or become less negative. That is, the IS^{**} curve becomes flatter.

Problem 5.12

a) i) At price level P^* – and thus at real wage $\bar{W}/P^*$ – employment and output are at their maximum possible levels. This is where the labor demand curve and the labor supply curve intersect and there is no unemployment.

a) ii) At a higher price, say $P' > P^*$ – and thus at a lower real wage $\bar{W}/P' < \bar{W}/P^*$ – labor demand exceeds labor supply. Given the "short-side" rule, this means that employment is determined by labor supply and is at L' in the diagram. Since $F'(L) > 0$, output is lower at L' (and P') than it is at L^* (and P^*).

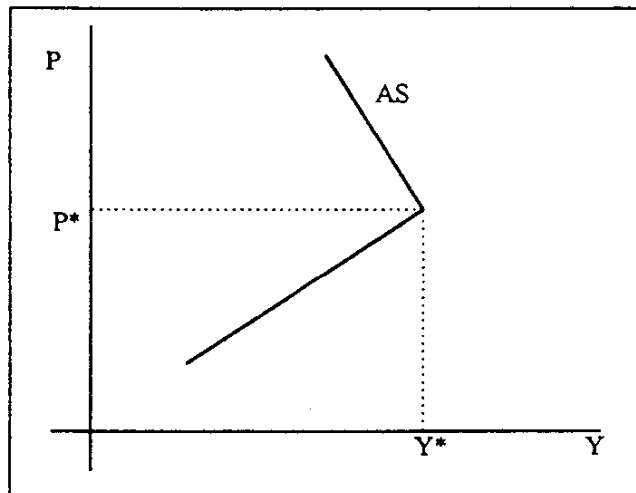


b) The above information allows us to answer part (b). As the price level rises toward P^* , employment is determined by labor demand. Thus employment and output rise as the economy moves down the labor demand curve.

At price P^* , output is at Y^* , its maximum possible value.

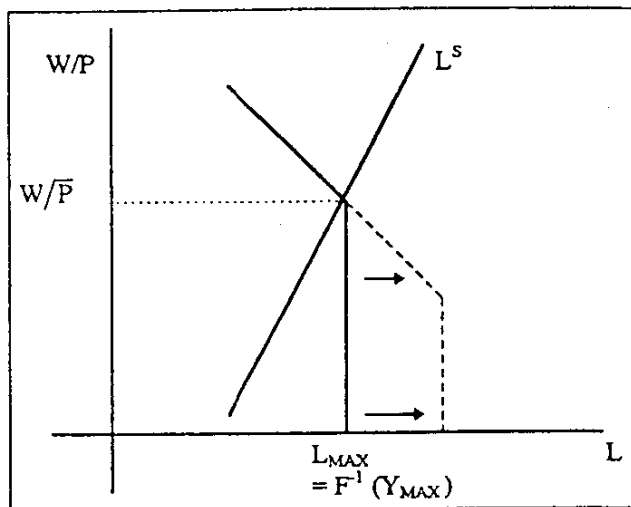
As the price level rises above P^* , employment is determined by labor supply. Thus employment and output fall as the economy moves down the labor supply curve. See the diagram for what this implies about the shape of the aggregate supply curve.

At prices above P^* , the AS curve is backward bending under the "short-side" assumption.



Problem 5.13

For this to be the case, the effective labor demand curve must intersect the labor supply curve right at the point at which it becomes vertical. Any further rightward shift in the effective labor demand curve would not cause the point of intersection of the two curves to change. See the diagram. Thus even if the effective labor demand curve shifts out, employment does not rise. Thus output cannot rise above Y_{MAX} .

**Problem 5.14**

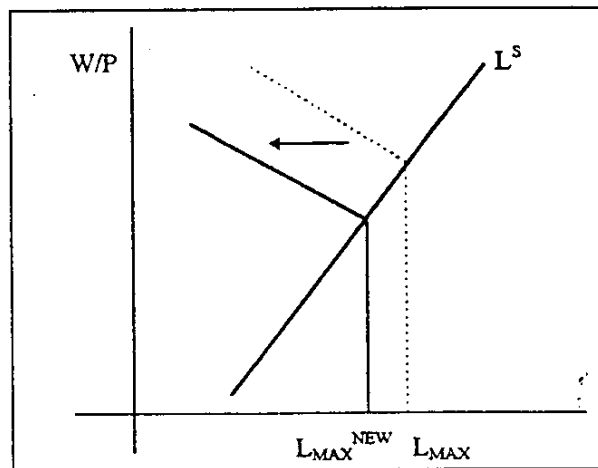
In Model 1 – Keynes's Model – firms hire labor up to the point where the marginal product of labor equals the real wage. That is, L is determined by:

$$(1) \quad AF'(L) = \bar{W}/P$$

At a given price – and thus at a given real wage – the level of L that equates the marginal product with the real wage is now lower after the fall in A . The labor demand curve shifts to the left. Firms reduce the amount of labor they demand at a given price. So output supplied by firms at a given price level is now lower for two reasons. A given amount of labor now produces less output and firms choose to hire less labor at a given P . Thus the AS curve shifts to the left.

In Model 2 – sticky prices, flexible wages and a competitive labor market – the AS curve is still horizontal at $P = \bar{P}$ out to Y_{MAX} , where Y_{MAX} is the level of output at which marginal cost just equals the fixed price level. However, Y_{MAX} itself will now change.

As described in the answers to Problem 5.13, at Y_{MAX} , the effective labor demand curve intersects the labor supply curve right at the point where it becomes vertical. Now with A lower, the downward sloping portion of the labor demand curve shifts to the left – at a given real wage, the level of L that equates the marginal product of labor and the real wage is lower. This means that there is a new lower L_{MAX} . The maximum amount of labor which it is profitable to hire to meet demand at the fixed price is now lower.



Therefore Y_{MAX} is now lower for two reasons.

The maximum amount of labor that it is profitable to hire at the fixed price is lower and a given amount of labor now produces less output.

In Model 3 – sticky prices, flexible wages and real labor market imperfections – we get essentially the same result as in Model 2. The level of output at which marginal cost just equals the fixed price is lower after the fall in A . Thus the AS curve is horizontal but out to a lower Y_{MAX} .

In Model 4 – sticky wages, flexible prices and imperfect competition – the price is set as a markup over marginal cost:

$$(2) \quad P = \mu(L) \frac{W}{AF'(L)}$$

A fall in A raises marginal cost at a given L . Assuming the markup does not depend on A , the price level rises for a given L . The AS curve shifts up, regardless of whether it is downward sloping, upward sloping or horizontal.

Problem 5.15

a) i) We can solve for output and the interest rate. Begin by substituting the expression for inflation – $\dot{p} = \theta y$ – into the goods market equilibrium relationship – $y = -a(i - \dot{p})$ – to obtain:

$$(1) \quad y(t) = -a[i(t) - \theta y(t)]$$

Rearranging the money market equilibrium condition – $m - p = -ki$ – yields:

$$(2) \quad i(t) = [p(t) - m(t)]/k$$

Substitute equation (2) into equation (1) to obtain:

$$y(t) = -a[(p(t) - m(t))/k - \theta y(t)] \Rightarrow y(t) - a\theta y(t) = (a/k) \cdot [m(t) - p(t)]$$

Simplifying and solving for output yields:

$$(3) \quad y(t) = \frac{a[m(t) - p(t)]}{k(1 - a\theta)}$$

Substituting $p(0) = 0$ and $m(0) = m'$ into equation (3) yields:

$$(4) \quad y(0) = \frac{a \cdot m'}{k(1 - a\theta)} < 0 \quad \text{since } a\theta < 1, a > 0, k > 0 \text{ and } m' < 0$$

Substituting $p(0) = 0$ and $m(0) = m'$ into equation (2) yields:

$$(5) \quad i(0) = -m'/k > 0$$

Thus the level of output falls from its previous value of 0, while the nominal interest rate rises from its previous value of 0.

In order to see how an increase in θ , the speed of price adjustment, affects the value of $y(0)$, take the derivative of output at time 0 with respect to θ :

$$\frac{\partial y(0)}{\partial \theta} = \frac{-a \cdot m' \cdot (-a)}{k(1 - a\theta)^2} = \frac{a^2 \cdot m'}{k(1 - a\theta)^2} < 0 \quad \text{since } m' < 0$$

Thus $y(0)$ falls even more from its initial value of 0 if θ is higher. The intuition is that a bigger θ – from $\dot{p} = \theta y$ – implies that as y falls there will be an even bigger drop in inflation. In turn, this means a bigger increase in the real interest rate. This reduces planned expenditure – and thus equilibrium output – even further.

a) ii) Take the time derivative of equation (3), noting that $m(t) = m'$ for all $t \geq 0$:

$$(6) \quad \dot{y}(t) = \frac{-a\dot{p}(t)}{k(1 - a\theta)}$$

Substituting the expression for inflation – $\dot{p} = \theta y$ – into equation (6) yields:

$$(7) \dot{y}(t) = \frac{-a\theta}{k(1-a\theta)} \cdot y(t)$$

So log output will exponentially approach its original value of 0. More formally, solving the differential equation (7) yields:

$$(8) y(t) = e^{[-a\theta/(1-a\theta)k] \cdot t} y(0)$$

Substituting equation (4) into equation (8) yields:

$$(9) y(t) = e^{[-a\theta/(1-a\theta)k] \cdot t} \cdot \frac{a \cdot m'}{k(1-a\theta)}$$

b) Substituting the expression for $y(t)$ from equation (9) into the definition of the amount of output

volatility caused by a disturbance -- $V = \int_{t=0}^{\infty} y(t)^2 dt$ -- yields:

$$V = \int_{t=0}^{\infty} e^{[-2a\theta/(1-a\theta)k] \cdot t} \cdot \left[\frac{am'}{k(1-a\theta)} \right]^2 dt \Rightarrow V = \left[\frac{am'}{k(1-a\theta)} \right]^2 \cdot \int_{t=0}^{\infty} e^{[-2a\theta/(1-a\theta)k] \cdot t} dt$$

Solving the integral is straightforward and doing so yields:

$$V = \left[\frac{am'}{k(1-a\theta)} \right]^2 \cdot \left[\frac{(1-a\theta)k}{2a\theta} \right]$$

Simplifying this expression allows us to obtain:

$$(10) V = \frac{am'^2}{2k(\theta - a\theta^2)}$$

To see how a change in the speed of price adjustment, θ , affects volatility, take the derivative of V with respect to θ :

$$(11) \frac{\partial V}{\partial \theta} = \frac{-am'^2}{2k(\theta - a\theta^2)^2} \cdot (1 - 2a\theta)$$

The sign of this derivative will be determined by the sign of $(1 - 2a\theta)$.

If $(1 - 2a\theta) > 0$ or $\theta < 1/2a$ then $\partial V/\partial \theta < 0$. Thus for "small" values of θ , a marginal increase in price flexibility will reduce output volatility. If $(1 - 2a\theta) < 0$ or $1/a > \theta > 1/2a$ then $\partial V/\partial \theta > 0$. Thus for "large" values of θ , a marginal increase in price flexibility will actually increase output volatility. [Note that we assumed from the outset that $a\theta < 1$ or equivalently $\theta < 1/a$.] Finally, note that the higher is a -- where a captures how responsive planned expenditure is to changes in the real interest rate -- the smaller is the range of θ 's over which increased price flexibility reduces output volatility.

SOLUTIONS TO CHAPTER 6

Problem 6.1

a) The individual's problem is to choose labor supply, L_i , to maximize expected utility, conditional on the realization of P_i . That is, the problem is:

$$\max_{L_i} E\left[\left(C_i - (1/\gamma) \cdot L_i^\gamma\right) \middle| P_i\right]$$

Substituting $C_i = P_i \cdot Q_i / P$ and $Q_i = L_i$ gives us:

$$\max_{L_i} E\left[\left(\frac{P_i L_i}{P} - \frac{1}{\gamma} \cdot L_i^\gamma\right) \middle| P_i\right]$$

Since only P is uncertain, this can be rewritten as:

$$\max_{L_i} E\left[\left(P_i / P\right) \middle| P_i\right] \cdot L_i - (1/\gamma) L_i^\gamma$$

The first-order condition is given by:

$$(1) \quad E[(P_i / P) | P_i] \cdot L_i^{\gamma-1} = 0$$

or:

$$L_i^{\gamma-1} = E[(P_i / P) | P_i]$$

Thus optimal labor supply is given by:

$$(2) \quad L_i = \left\{ E[(P_i / P) | P_i] \right\}^{1/(\gamma-1)}$$

Taking the log of both sides of equation (2) and defining $\ell_i \equiv \ln L_i$ yields:

$$(3) \quad \ell_i = [1/(\gamma - 1)] \cdot \ln E[(P_i / P) | P_i]$$

b) The amount of labor the individual supplies if she follows the certainty-equivalence rule is given by (in logs):

$$(4) \quad \ell_i = [1/(\gamma - 1)] \cdot E[\ln(P_i / P) | P_i]$$

Since $\ln(P_i / P)$ is a concave function of (P_i / P) , then by Jensen's inequality $\ln E[(P_i / P) | P_i] > E[\ln(P_i / P) | P_i]$

Thus the amount of labor the individual supplies if she follows the certainty-equivalence rule is less than the optimal amount derived in part (a).

c) We are given that:

$$(5) \quad \ln(P_i / P) = E[\ln(P_i / P) | P_i] + u_i \quad u_i \sim N(0, V_u)$$

Taking the exponential function of both sides of equation (5) yields:

$$(6) \quad P_i / P = e^{E[\ln(P_i / P) | P_i]} \cdot e^{u_i}$$

Now take the expected value, conditional on P_i , of both sides of equation (6):

$$(7) \quad E[(P_i / P) | P_i] = e^{E[\ln(P_i / P) | P_i]} \cdot E(e^{u_i} | P_i)$$

Taking the natural log of both sides of equation (7) yields:

$$(8) \quad \ln E[(P_i / P) | P_i] = E[\ln(P_i / P) | P_i] + \ln E(e^{u_i} | P_i)$$

Note that $\ln E(e^{u_i} | P_i)$ is just a constant whose value is independent of P_i . Substituting equation (8) into equation (3) -- the optimal amount of (log) labor supply -- gives us:

$$\ell_i = [1/(\gamma - 1)] \cdot \left\{ E[\ln(P_i / P) | P_i] + \ln E(e^{u_i} | P_i) \right\}$$

or simply:

$$(9) \quad \ell_i = [1/(\gamma - 1)] \cdot E[\ln(P_i/P)|P_i] + [1/(\gamma - 1)] \cdot \left[\ln E(e^{u_i} | P_i) \right]$$

The first term $- [1/(\gamma - 1)] \cdot E[\ln(P_i/P)|P_i]$ is the certainty-equivalence choice of (log) labor supply, while the second term is a constant. Thus the ℓ_i that maximizes expected utility differs from the certainty-equivalence rule only by a constant.

Problem 6.2

a) The individual's problem is to maximize $C_i = \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{\eta/(\eta-1)}$ subject to the budget

constraint, $\int_{j=0}^1 P_j C_{ij} dj = Y_i$. Set up the Lagrangian:

$$(1) \quad \mathcal{L} = \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{\eta/(\eta-1)} + \lambda \left[Y_i - \int_{j=0}^1 P_j C_{ij} dj \right]$$

The first-order condition for a representative good, C_{ij} , is:

$$\frac{\partial \mathcal{L}}{\partial C_{ij}} = \left(\frac{\eta}{\eta-1} \right) \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{[\eta/(\eta-1)]-1} \left(\frac{\eta-1}{\eta} \right) Z_j^{1/\eta} C_{ij}^{[(\eta-1)/\eta]-1} - \lambda P_j = 0$$

We can simplify the exponents since $[\eta/(\eta-1)] - 1 = 1/(\eta-1)$ and $[(\eta-1)/\eta] - 1 = -1/\eta$. Thus the preceding expression implies:

$$(2) \quad C_{ij}^{-1/\eta} = \frac{\lambda P_j}{Z_j^{1/\eta} \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{1/(\eta-1)}}$$

Taking both sides of equation (2) to the exponent $-\eta$ yields:

$$(3) \quad C_{ij} = \frac{Z_j \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{\eta/(\eta-1)}}{(\lambda P_j)^\eta}$$

b) For each good on the unit interval, there will be an equation like (3). Thus for some other good, C_{ik} , we can write:

$$(4) \quad C_{ik} = \frac{Z_k \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{\eta/(\eta-1)}}{(\lambda P_k)^\eta}$$

Dividing equation (3) by equation (4) yields:

$$\frac{C_{ij}}{C_{ik}} = \frac{Z_j \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{\eta/(\eta-1)}}{Z_k \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{\eta/(\eta-1)}} \frac{(\lambda P_k)^\eta}{(\lambda P_j)^\eta}$$

Simplifying yields:

$$(5) C_{ij}/C_{ik} = (P_k/P_j)^\eta (Z_j/Z_k)$$

Writing C_{ij} in terms of C_{ik} , we have:

$$(6) C_{ij} = (P_k/P_j)^\eta (Z_j/Z_k) C_{ik}$$

Substituting equation (6) into the budget constraint, $\int_{j=0}^1 P_j C_{ij} dj = Y_i$, yields:

$$\int_{j=0}^1 P_j (P_k/P_j)^\eta (Z_j/Z_k) C_{ik} dj = Y_i$$

Pulling the terms not indexed by j out of the integral sign leaves us with:

$$\frac{C_{ik} P_k^\eta}{Z_k} \int_{j=0}^1 Z_j P_j^{1-\eta} dj = Y_i$$

and then solving for C_{ik} yields:

$$(7) C_{ik} = \frac{Y_i Z_k}{P_k^\eta \left[\int_{j=0}^1 Z_j P_j^{1-\eta} dj \right]}$$

Equation (7) holds for all goods. Thus returning to the notation of C_{ij} as our representative good gives us:

$$(8) C_{ij} = \frac{Y_i Z_j}{P_j^\eta \left[\int_{j=0}^1 Z_j P_j^{1-\eta} dj \right]}$$

c) Substituting equation (8) into the expression for $C_i - C_i = \left[\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj \right]^{\eta/(\eta-1)}$ -- yields:

$$C_i = \left[\int_{j=0}^1 \frac{Z_j^{1/\eta} Y_i^{(\eta-1)/\eta} Z_j^{(\eta-1)/\eta}}{P_j^{(\eta-1)} \left[\int_{j=0}^1 Z_j P_j^{1-\eta} dj \right]^{(\eta-1)/\eta}} dj \right]^{\eta/(\eta-1)}$$

Pulling the terms not indexed by j out of the integral gives us:

$$C_i = \left[\frac{Y_i^{(\eta-1)/\eta}}{\left[\int_{j=0}^1 Z_j P_j^{1-\eta} dj \right]^{(\eta-1)/\eta}} \left(\int_{j=0}^1 Z_j P_j^{1-\eta} dj \right) \right]^{\eta/(\eta-1)}$$

This simplifies to:

$$(9) C_i = Y_i \left[\int_{j=0}^1 Z_j P_j^{1-\eta} dj \right]^{[\eta/(\eta-1)]-1}$$

Since $[\eta/(\eta-1)] - 1 = 1/(\eta-1) = -1/(1-\eta)$, equation (9) can be rewritten as:

$$(10) C_i = \frac{Y_i}{\left[\sum_{j=0}^1 Z_j P_j^{1-\eta} d_j \right]^{1/(1-\eta)}}$$

Defining the price index as:

$$(11) P \equiv \left[\sum_{j=0}^1 Z_j P_j^{1-\eta} d_j \right]^{1/(1-\eta)}$$

we have:

$$(12) C_i = Y_i / P$$

d) Note that $\sum_{j=0}^1 Z_j P_j^{1-\eta} d_j \equiv P^{(1-\eta)}$. Using this fact, equation (8) -- individual i's demand for good j -- can be rewritten as:

$$(13) C_{ij} = \frac{Y_i Z_j}{P_j^\eta P^{1-\eta}}$$

This can be rearranged to yield:

$$(14) C_{ij} = Z_j \left(P_j / P \right)^{-\eta} (Y_i / P)$$

Equation (14) gives individual i's demand for good j as a function of the taste shock for good j, good j's relative price and individual i's real income.

e) Taking the log of both sides of equation (14) yields:

$$(15) c_{ij} = z_j - \eta(p_j - p) + (y_i - p)$$

This resembles equation (6.7) in the text. However, the price index given in equation (11) is not simply the average of the individual p's, as it is in equation (6.9).

Problem 6.3

a) Model (i) is given by:

$$(1) y_t = a' z_{t-1} + b e_t + v_t$$

This model says that only the unexpected component of money, e_t , affects output. Model (ii) is given by:

$$(2) y_t = \alpha' z_{t-1} + \beta m_t + v_t$$

This model says that all money matters for output.

Substituting the assumption about monetary policy -- $m_t = c' z_{t-1} + e_t$ -- into equation (2) yields:

$$(3) y_t = \alpha' z_{t-1} + \beta [c' z_{t-1} + e_t] + v_t$$

or collecting the terms in z_{t-1} :

$$(4) y_t = (\alpha' + \beta c') z_{t-1} + \beta e_t + v_t$$

The models given by equations (1) and (4) cannot be distinguished. Given some a' and b , $\alpha' = a' - \beta c'$ and $\beta = b$ have the same predictions. Intuitively, it is not possible to separate the direct effect of the z 's on output from any possible indirect effect they may have through monetary policy. So it could be the case that only unexpected money matters and the effect of the z 's on output that we observe is simply their direct effect. However, it could also be the case that the expected component of money affects output and thus the effect of the z 's that we observe consists of both the direct and indirect effects.

b) Substituting the new assumption about monetary policy -- $m_t = c' z_{t-1} + \gamma' w_{t-1} + e_t$ -- into model (ii) yields:

$$(5) y_t = \alpha' z_{t-1} + \beta[c' z_{t-1} + \gamma' w_{t-1} + e_t] + v_t$$

or collecting the z_{t-1} terms:

$$(6) y_t = (\alpha' + \beta c') z_{t-1} + \beta \gamma' w_{t-1} + \beta e_t + v_t$$

In this case, it is possible to distinguish between the two theories. Model (i) -- only unexpected money matters -- predicts that the coefficients on the w 's should be zero. Model (ii) -- all money matters -- does not predict this.

Problem 6.4

Using the correct price index does not alter the analysis of the individual's behavior. That is, equation (6.40) in the text, which defines the optimal relative price of individual i 's good as:

$$(1) \frac{P_i}{P} = \frac{\eta}{\eta - 1} \frac{W}{P}$$

still holds. Similarly, individual i 's optimal choice of labor supply is unaffected. It is still given by equation (6.42) in the text, or:

$$(2) L_i = \left(\frac{W}{P} \right)^{1/(\gamma-1)}$$

We need to solve for equilibrium output, Y , and the price level, P , using the fact that total spending in the economy equals M , or:

$$(3) \int_{i=0}^1 P_i Q_i di = M$$

where Q_i equals output of good i . Since the production function is $Q_i = L_i$, output of good i is:

$$(4) Q_i = \left(\frac{W}{P} \right)^{1/(\gamma-1)}$$

Multiplying both sides of equation (1) by P gives us:

$$(5) P_i = \frac{\eta}{\eta - 1} W$$

From equation (11) in Problem 6.2, the price index -- with all the Z_j 's equal to zero -- is given by:

$$(6) P = \left[\int_{i=0}^1 P_i^{1-\eta} di \right]^{1/(1-\eta)}$$

Substituting equation (5) into the price index given by equation (6) yields:

$$(7) P = \left[\int_{i=0}^1 \left(\frac{\eta}{\eta - 1} W \right)^{1-\eta} di \right]^{1/(1-\eta)} = \left[\left(\frac{\eta}{\eta - 1} W \right)^{1-\eta} \int_{i=0}^1 di \right]^{1/(1-\eta)}$$

which implies:

$$(8) P = \frac{\eta}{\eta - 1} W$$

Rearranging equation (8) gives us the equilibrium real wage:

$$(9) \frac{W}{P} = \frac{\eta - 1}{\eta}$$

Substituting equation (9) into equation (4) gives individual i 's output:

$$(10) Q_i = \left(\frac{\eta - 1}{\eta} \right)^{1/(\gamma-1)}$$

Substituting $P_i = P$ and equation (10) into equation (3) yields:

$$(11) \int_{i=0}^1 P \left(\frac{\eta - 1}{\eta} \right)^{1/(\gamma-1)} di = M$$

Since nothing inside the integral is indexed by i , we have:

$$(12) P \left(\frac{\eta - 1}{\eta} \right)^{1/(\gamma-1)} \int_{i=0}^1 di = M$$

Solving equation (12) for P gives us:

$$(13) P = \left(\frac{\eta - 1}{\eta} \right)^{-1/(\gamma-1)} M = \frac{M}{[(\eta - 1)/\eta]^{1/(\gamma-1)}}$$

Equation (13) is identical to equation (6.47) in the text.

Finally, since aggregate demand is given by $Y = M/P$, equilibrium output is:

$$(14) Y = \frac{M}{M/[(\eta - 1)/\eta]^{1/(\gamma-1)}} = \left(\frac{\eta - 1}{\eta} \right)^{1/(\gamma-1)}$$

Equation (14) is identical to equation (6.46) in the text. Thus using the correct price index does not affect the expressions for equilibrium price and output.

Problem 6.5

a) Substituting the expression for the nominal wage — $w = \theta p$ — into the aggregate price equation — $p = w + (1 - \alpha)\ell - s$ — yields $p = \theta p + (1 - \alpha)\ell - s$. Solving for p yields:

$$(1) p = [(1 - \alpha)\ell - s]/(1 - \theta)$$

Substituting the aggregate output equation — $y = s + \alpha\ell$ — and equation (1) for the price level into the aggregate demand equation — $y = m - p$ — yields:

$$s + \alpha\ell = m - [(1 - \alpha)\ell - s]/(1 - \theta)$$

Collecting the terms in ℓ leaves us with:

$$\alpha\ell + [(1 - \alpha)\ell/(1 - \theta)] = m + [s/(1 - \theta)] - s$$

Obtaining a common denominator and simplifying gives us:

$$[\alpha(1 - \theta) + (1 - \alpha)]\ell/(1 - \theta) = m + [1 - (1 - \theta)]s/(1 - \theta)$$

or:

$$(1 - \alpha\theta)\ell/(1 - \theta) = m + [\theta s/(1 - \theta)]$$

and thus finally, employment is given by:

$$(2) \ell = \frac{(1 - \theta)m + \theta s}{(1 - \alpha\theta)}$$

Substituting equation (2) into equation (1) yields:

$$p = \frac{(1 - \alpha)[(1 - \theta)m + \theta s]}{(1 - \theta)(1 - \alpha\theta)} - \frac{s}{(1 - \theta)}$$

Simplifying gives us:

$$p = \frac{(1-\alpha)(1-\theta)m + (1-\alpha)\theta s - (1-\alpha\theta)s}{(1-\theta)(1-\alpha\theta)} = \frac{(1-\alpha)(1-\theta)m - (1-\theta)s}{(1-\theta)(1-\alpha\theta)}$$

Thus, the aggregate price level is given by:

$$(3) \quad p = \frac{(1-\alpha)m - s}{(1-\alpha\theta)}$$

Substituting equation (2) into the aggregate output equation — $y = s + \alpha\ell$ — and simplifying yields:

$$y = s + \frac{\alpha(1-\theta)m + \alpha\theta s}{(1-\alpha\theta)} = \frac{s - \alpha\theta s + \alpha(1-\theta)m + \alpha\theta s}{(1-\alpha\theta)}$$

And therefore, output is given by:

$$(4) \quad y = \frac{s + \alpha(1-\theta)m}{(1-\alpha\theta)}$$

Finally, to get an expression for the nominal wage, substitute equation (3) into $w = \theta p$:

$$(5) \quad w = \frac{\theta[(1-\alpha)m - s]}{(1-\alpha\theta)}$$

The next step is to see how the degree of indexation affects the responsiveness of employment to monetary shocks. First, use equation (2) to find how employment varies with m :

$$(6) \quad \frac{\partial \ell}{\partial m} = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)}$$

Taking the derivative of both sides of equation (6) with respect to θ gives us:

$$(7) \quad \frac{\partial[\partial \ell / \partial m]}{\partial \theta} = \frac{(-1)[1-\alpha\theta] - (1-\theta)(-\alpha)}{(1-\alpha\theta)^2} = \frac{(\alpha-1)}{(1-\alpha\theta)^2} < 0$$

Thus an increase in the degree of indexation, θ , reduces the amount that employment will change due to a given monetary shock.

The next step is to see how the degree of indexation affects the responsiveness of employment to supply shocks. First, use equation (2) to find how employment varies with s :

$$(8) \quad \frac{\partial \ell}{\partial s} = \frac{\theta}{(1-\alpha\theta)}$$

Taking the derivative of both sides of equation (8) with respect to θ gives us:

$$(9) \quad \frac{\partial[\partial \ell / \partial s]}{\partial \theta} = \frac{(1)[1-\alpha\theta] - (\theta)(-\alpha)}{(1-\alpha\theta)^2} = \frac{1}{(1-\alpha\theta)^2} > 0$$

Thus an increase in the degree of wage indexation, θ , increases the amount that employment will change due to a given supply shock.

b) From equation (2), the variance of employment is given by:

$$(10) \quad V_{\ell} = \left[\frac{(1-\theta)}{(1-\alpha\theta)} \right]^2 V_m + \left[\frac{\theta}{(1-\alpha\theta)} \right]^2 V_s$$

where we have used the fact that m and s are independent random variables with variances V_m and V_s . We need to find the value of θ that minimizes this variance of employment. The first-order condition for this minimization is:

$$(11) \quad \frac{\partial V_\ell}{\partial \theta} = 2 \left[\frac{(1-\theta)}{(1-\alpha\theta)} \right] \left[\frac{(\alpha-1)}{(1-\alpha\theta)^2} \right] V_m + 2 \left[\frac{\theta}{(1-\alpha\theta)} \right] \left[\frac{1}{(1-\alpha\theta)^2} \right] V_s = 0$$

Equation (11) simplifies to:

$$(1-\theta)(\alpha-1)V_m + \theta V_s = 0$$

Collecting the terms in θ gives us:

$$\theta[(1-\alpha)V_m + V_s] = (1-\alpha)V_m$$

Thus the optimal degree of wage indexation is:

$$(12) \quad \theta = \frac{(1-\alpha)V_m}{(1-\alpha)V_m + V_s}$$

Given the result in part (a) — that indexation reduces the impact on employment of monetary shocks but increases the impact from supply shocks — equation (12) is intuitive. First of all, if $V_s = 0$ — so that there are no supply shocks — the optimal degree of indexation is one. In addition, the larger is the variance of the supply shocks relative to the variance of the monetary shocks, the lower is the optimal degree of indexation.

c) i) As stated in the problem:

$$(13) \quad y_i = y - \phi(w_i - w)$$

where $\phi \equiv \alpha\eta/[\alpha + (1-\alpha)\eta]$. Since $w = \theta p$ and $w_i = \theta_i p$, equation (13) becomes:

$$(14) \quad y_i = y - \phi(\theta_i p - \theta p) = y - (\theta_i - \theta)\phi p$$

From the production function, $y_i = s + \alpha\ell_i$ and $y = s + \alpha\ell$ and thus we can write:

$$(15) \quad y_i - y = \alpha(\ell_i - \ell)$$

Solving equation (15) for employment at firm i yields:

$$(16) \quad \ell_i = \ell + (1/\alpha)(y_i - y)$$

Substituting equation (14) for $y_i - y$ into equation (16) gives us:

$$(17) \quad \ell_i = \ell - (1/\alpha)(\theta_i - \theta)\phi p$$

Substituting equation (2) for aggregate employment and equation (3) for the price level into equation (17) gives us:

$$(18) \quad \ell_i = \frac{(1-\theta)m + \theta s}{(1-\alpha\theta)} - \frac{(\theta_i - \theta)\phi[(1-\alpha)m - s]}{\alpha(1-\alpha\theta)} = \frac{1}{\alpha(1-\alpha\theta)} [\alpha(1-\theta)m + \alpha\theta s - (\theta_i - \theta)\phi[(1-\alpha)m - s]]$$

which implies:

$$(19) \quad \ell_i = \frac{1}{\alpha(1-\alpha\theta)} \{ m[\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)] + s[\alpha\theta + (\theta_i - \theta)\phi] \}$$

c) ii) From equation (19), the variance of employment at firm i is given by:

$$(20) \quad \text{Var}(\ell_i) = \left[\frac{\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)}{\alpha(1-\alpha\theta)} \right]^2 V_m + \left[\frac{\alpha\theta + (\theta_i - \theta)\phi}{\alpha(1-\alpha\theta)} \right]^2 V_s$$

The first-order condition for the value of θ_i — the degree of wage indexation at firm i — that minimizes the variance of employment at firm i is:

$$(21) \quad \frac{\partial \text{Var}(\ell_i)}{\partial \theta_i} = 2 \left[\frac{\alpha(1-\theta) - (\theta_i - \theta)\phi(1-\alpha)}{\alpha(1-\alpha\theta)} \right] [-\phi(1-\alpha)] V_m + 2 \left[\frac{\alpha\theta + (\theta_i - \theta)\phi}{\alpha(1-\alpha\theta)} \right] \phi V_s = 0$$

Equation (21) simplifies to:

$$(22) \quad \{ \alpha(1-\theta) - \theta_i [\phi(1-\alpha)] + \theta\phi(1-\alpha) \} \phi(1-\alpha)V_m = (\alpha\theta + \theta_i\phi - \theta\phi)\phi V_s$$

which implies:

$$(23) \quad \theta_i \phi^2 V_s + \theta_i [\phi(1-\alpha)]^2 V_m = [\alpha(1-\theta) + \theta\phi(1-\alpha)] \phi(1-\alpha)V_m - (\alpha\theta - \theta\phi)\phi V_s$$

Thus θ_i is given by:

$$(24) \theta_i = \frac{[\alpha(1-\theta) + \theta\phi(1-\alpha)]\phi(1-\alpha)V_m - [\theta(\phi-\alpha)]\phi V_s}{\phi^2 V_s + [\phi(1-\alpha)]^2 V_m}$$

c) iii) We need to find a value of θ such that the first-order condition given by equation (22) holds when $\theta_i = \theta$. That is, we need to find a value of θ such that if economy-wide indexation is given by θ , the representative firm – in order to minimize its employment fluctuations – wishes to choose θ as well.

Setting $\theta_i = \theta$ in equation (22) gives us:

$$(25) \alpha(1-\theta)\phi(1-\alpha)V_m = \alpha\theta\phi V_s$$

which implies:

$$(26) \theta[V_s + (1-\alpha)V_m] = (1-\alpha)V_m$$

Thus the Nash-equilibrium value of θ is:

$$(27) \theta^{EQ} = \frac{(1-\alpha)V_m}{(1-\alpha)V_m + V_s}$$

This is exactly the same value of θ we found in part (b) – see equation (12). The value of θ that minimizes the variance of aggregate fluctuations in employment is also a Nash equilibrium. Given that other firms are choosing θ^{EQ} as their degree of wage indexation, it is optimal for any individual firm to choose θ^{EQ} as well.

Problem 6.6

a) The representative individual will set her price equal to the average of the optimal price for t and the expected optimal price for $t+1$. Thus:

$$(1) x_t = (p_t^* + E_t p_{t+1}^*)/2$$

Since $p_t^* = \phi m_t + (1-\phi)p_t$ and this holds for all periods, we have:

$$(2) x_t = [(\phi m_t + (1-\phi)p_t) + (\phi E_t m_{t+1} + (1-\phi)E_t p_{t+1})]/2$$

b) With synchronization, $p_t = x_t$ and $p_{t+1} = x_t$ and thus:

$$(3) x_t = [(\phi m_t + (1-\phi)x_t) + (\phi E_t m_{t+1} + (1-\phi)x_t)]/2$$

Simplifying yields:

$$x_t = [2(1-\phi)x_t + \phi(m_t + E_t m_{t+1})]/2 \Rightarrow [1 - (1-\phi)]x_t = \phi(m_t + E_t m_{t+1})/2$$

and thus:

$$(4) x_t = (m_t + E_t m_{t+1})/2$$

Firms set their price equal to the average of this period's value of m and the expected value of next period's value of m .

c) Substitute $p_t = x_t = (m_t + E_t m_{t+1})/2$ into the aggregate demand equation to obtain:

$$y_t = m_t - (m_t + E_t m_{t+1})/2$$

Simplifying gives us:

$$(5) y_t = (m_t - E_t m_{t+1})/2$$

Assuming that m follows a random walk so that $E_t m_{t+1} = m_t$, we have:

$$y_t = (m_t - m_t)/2$$

or simply:

$$(6) y_t = 0$$

Now, substituting $p_{t+1} = x_t = (m_t + E_t m_{t+1})/2$ into the aggregate demand equation for period $t+1$ gives us:

$$y_{t+1} = m_{t+1} - [(m_t + E_t m_{t+1})/2] = (m_{t+1} - E_t m_{t+1}) - [(m_t - E_t m_{t+1})/2]$$

Assuming that m follows a random walk so that $E_t m_{t+1} = m_t$, we have:

$$y_{t+1} = m_{t+1} - (m_t + m_t)/2$$

or simply:

$$(7) y_{t+1} = m_{t+1} - m_t$$

The central result of the Taylor model does not continue to hold. Nominal disturbances that occur in periods when firms are not setting prices feed through one-for-one into output – see equation (7). However, once firms set prices again, output returns to its normal value of 0 – see equation (6).

Intuitively, we have removed the mechanism by which the Taylor model generates long-lasting effects of nominal disturbances. In the Taylor model, once firms get to adjust price, they do not adjust fully to a nominal disturbance because they know that not all firms are adjusting at that time. Thus fully adjusting will change their relative price, which they are reluctant to do. But here, firms know that all firms are also adjusting their price at the same time. Thus firms real prices will not change if they fully adjust and thus they do so.

Problem 6.7

When t is even, the price level is given by:

$$(1) p_t = f p_t^1 + (1 - f) p_t^2$$

where p_t^1 denotes the price set for t by individuals who set their prices in $t - 1$ – which was an odd period and hence fraction f of firms set p_t^1 – while p_t^2 denotes the price set for t by individuals setting prices in $t - 2$ – which was an even period and hence fraction $(1 - f)$ of firms set p_t^2 . Now, p_t^1 equals the expectation as of period $t - 1$ of p_t^* and thus:

$$(2) p_t^1 = E_{t-1} p_t^* = E_{t-1} [\phi m_t + (1 - \phi) p_t]$$

Substituting equation (1) into equation (2) and using the fact that p_t^2 is already known when p_t^1 is set and is thus not uncertain, yields:

$$(3) p_t^1 = \phi E_{t-1} m_t + (1 - \phi) [f p_t^1 + (1 - f) p_t^2]$$

Some simple algebra allows us to solve for p_t^1 :

$$(4) p_t^1 = \frac{\phi}{1 - (1 - \phi)f} E_{t-1} m_t + \frac{(1 - \phi)(1 - f)}{1 - (1 - \phi)f} p_t^2$$

Now, p_t^2 equals the expectation as of period $t - 2$ of p_t^* and thus:

$$(5) p_t^2 = E_{t-2} p_t^* = E_{t-2} [\phi m_t + (1 - \phi) p_t]$$

Substituting equation (1) into equation (5) yields:

$$(6) p_t^2 = \phi E_{t-2} m_t + (1 - \phi) [f E_{t-2} p_t^1 + (1 - f) p_t^2]$$

We need to find $E_{t-2} p_t^1$. Since the left and right-hand sides of equation (4) are equal and since expectations are rational, the expectation as of period $t - 2$ of these two expressions must be equal. That is, we have:

$$(7) E_{t-2} p_t^1 = \frac{\phi}{1 - (1 - \phi)f} E_{t-2} m_t + \frac{(1 - \phi)(1 - f)}{1 - (1 - \phi)f} p_t^2$$

where we have used the law of iterated projections so that $E_{t-2} E_{t-1} m_t = E_{t-2} m_t$. Substituting equation (7) into equation (6) yields:

$$p_t^2 = \phi E_{t-2} m_t + (1 - \phi) \left[\frac{\phi f}{1 - (1 - \phi)f} E_{t-2} m_t + \frac{(1 - \phi)(1 - f)f}{1 - (1 - \phi)f} p_t^2 + (1 - f) p_t^2 \right]$$

Collecting terms gives us:

$$p_t^2 = \left[\frac{\phi - \phi(1 - \phi)f + (1 - \phi)\phi f}{1 - (1 - \phi)f} \right] E_{t-2} m_t + (1 - \phi) \left[\frac{(1 - \phi)(1 - f)f + (1 - f) - (1 - \phi)(1 - f)f}{1 - (1 - \phi)f} \right] p_t^2$$

which simplifies to:

$$p_t^2 = \frac{\phi}{1 - (1 - \phi)f} E_{t-2} m_t + \frac{(1 - \phi)(1 - f)}{1 - (1 - \phi)f} p_t^2$$

Collecting the terms in p_t^2 gives us:

$$\left[\frac{1 - (1 - \phi)f - (1 - \phi)(1 - f)}{1 - (1 - \phi)f} \right] p_t^2 = \frac{\phi}{1 - (1 - \phi)f} E_{t-2} m_t$$

or simply:

$$\frac{\phi}{1 - (1 - \phi)f} p_t^2 = \frac{\phi}{1 - (1 - \phi)f} E_{t-2} m_t$$

And thus the price set in period $t - 2$ is given by:

$$(8) \quad p_t^2 = E_{t-2} m_t$$

Now to solve for the price set for t by those setting price in $t - 1$, substitute equation (8) into equation (4):

$$(9) \quad p_t^1 = \frac{\phi}{1 - (1 - \phi)f} E_{t-1} m_t + \frac{(1 - \phi)(1 - f)}{1 - (1 - \phi)f} E_{t-2} m_t$$

Adding and subtracting $E_{t-2} m_t$ to the right-hand side of equation (9) yields:

$$(10) \quad p_t^1 = E_{t-2} m_t + \frac{\phi}{1 - (1 - \phi)f} E_{t-1} m_t + \left[\frac{(1 - \phi)(1 - f) - 1 + (1 - \phi)f}{1 - (1 - \phi)f} \right] E_{t-2} m_t$$

Since $(1 - \phi)(1 - f) - 1 + (1 - \phi)f = -(1 - \phi)f + (1 - \phi) - 1 + (1 - \phi)f = -\phi$, equation (10) can be rewritten as:

$$(11) \quad p_t^1 = E_{t-2} m_t + \frac{\phi}{1 - (1 - \phi)f} (E_{t-1} m_t - E_{t-2} m_t)$$

To get an expression for the aggregate price level, substitute the formulas for p_t^1 and p_t^2 — equations (11) and (8) — into equation (1):

$$p_t = f \left[E_{t-2} m_t + \frac{\phi}{1 - (1 - \phi)f} (E_{t-1} m_t - E_{t-2} m_t) \right] + (1 - f) E_{t-2} m_t$$

Simplifying yields:

$$(12) \quad p_t = E_{t-2} m_t + \frac{\phi f}{1 - (1 - \phi)f} (E_{t-1} m_t - E_{t-2} m_t)$$

To solve for output in period t , substitute equation (12) into the expression for aggregate demand, $y_t = m_t - p_t$:

$$(13) \quad y_t = m_t - E_{t-2} m_t - \frac{\phi f}{1 - (1 - \phi)f} (E_{t-1} m_t - E_{t-2} m_t)$$

Collecting the terms in $E_{t-2} m_t$ as well as adding and subtracting $E_{t-1} m_t$ to the right-hand side of equation (13) gives us:

$$(14) \quad y_t = m_t - E_{t-1} m_t + \left[\frac{\phi f - 1 + (1 - \phi)f}{1 - (1 - \phi)f} \right] E_{t-2} m_t + \left[\frac{1 - (1 - \phi)f - \phi f}{1 - (1 - \phi)f} \right] E_{t-1} m_t$$

Since $\phi f - 1 + (1 - \phi)f = -(1 - f)$ and $1 - (1 - \phi)f - \phi f = (1 - f)$, equation (14) can be rewritten as:

$$(15) \quad y_t = \frac{(1 - f)}{1 - (1 - \phi)f} (E_{t-1} m_t - E_{t-2} m_t) + (m_t - E_{t-1} m_t)$$

Equations (12) and (15) give equilibrium price and output for an even period.

The analysis for the case of t odd is identical to the preceding analysis, except that the roles of f and $(1 - f)$ are reversed. Thus, derivations analogous to those used to obtain (12) and (15) yield:

$$(16) \quad p_t = E_{t-2} m_t + \frac{\phi(1 - f)}{1 - (1 - \phi)(1 - f)} (E_{t-1} m_t - E_{t-2} m_t)$$

and:

$$(17) \quad y_t = \frac{f}{1 - (1 - \phi)(1 - f)} (E_{t-1} m_t - E_{t-2} m_t) + (m_t - E_{t-1} m_t)$$

Equations (16) and (17) give equilibrium price and output for an odd period.

Problem 6.8

We will deal first with firms that set price in odd periods. Suppose that period t is an even period. Then p_{it} is the price set for an even period by a firm that sets prices in odd periods, while p_{it+1} is the price set for an odd period by a firm that sets prices in odd periods. From equation (11) in Problem 6.7, p_{it} is given by:

$$(1) \quad p_{it} = E_{t-2} m_t + \frac{\phi}{1 - (1 - \phi)f} (E_{t-1} m_t - E_{t-2} m_t)$$

With the assumption that m is a random walk, we have $E_{t-2} m_t = m_{t-2}$ and $E_{t-1} m_t = m_{t-1}$. Thus:

$$(2) \quad p_{it} = m_{t-2} + \frac{\phi}{1 - (1 - \phi)f} (m_{t-1} - m_{t-2})$$

As usual, the optimal price for period t is given by:

$$(3) \quad p_{it}^* = \phi m_t + (1 - \phi)p_t$$

From equation (12) in Problem 6.7, the aggregate price level in an even period is:

$$(4) \quad p_t = E_{t-2} m_t + \frac{\phi f}{1 - (1 - \phi)f} (E_{t-1} m_t - E_{t-2} m_t)$$

Again, since m follows a random walk, this is equivalent to:

$$(5) \quad p_t = m_{t-2} + \frac{\phi f}{1 - (1 - \phi)f} (m_{t-1} - m_{t-2})$$

Substituting equation (5) into equation (3) yields the optimal price in period t :

$$(6) \quad p_{it}^* = \phi m_t + (1 - \phi)m_{t-2} + \frac{(1 - \phi)\phi f}{1 - (1 - \phi)f} (m_{t-1} - m_{t-2})$$

Thus the amount of profit a firm expects to lose in period t , $E_t (p_{it} - p_{it}^*)^2$, is proportional to:

$$E_t \left[m_{t-2} + \frac{\phi}{1 - (1 - \phi)f} (m_{t-1} - m_{t-2}) - \phi m_t - (1 - \phi)m_{t-2} + \frac{(1 - \phi)\phi f}{1 - (1 - \phi)f} (m_{t-1} - m_{t-2}) \right]^2$$

Collecting terms yields:

$$(7) \quad E_t (p_{it} - p_{it}^*)^2 = E_t \left[-\phi m_t + \frac{\phi[1 - (1 - \phi)f]}{1 - (1 - \phi)f} (m_{t-1} - m_{t-2}) + \phi m_{t-2} \right]^2$$

Simplifying gives us:

$$E_t (p_{it} - p_{it}^*)^2 = E_t [-\phi m_t + \phi m_{t-1} - \phi m_{t-2} + \phi m_{t-2}]^2$$

or:

$$(8) \quad E_t (p_{it} - p_{it}^*)^2 = \phi^2 E_t (m_{t-1} - m_t)^2$$

Now, the price set for an odd period by a firm that sets prices in odd periods is given by:

$$(9) \quad p_{it+1} = E_{t-1} m_{t+1}$$

Since m follows a random walk, $E_{t-1} m_{t+1} = m_{t-1}$ and thus:

$$(10) \quad p_{it+1} = m_{t-1}$$

The optimal price for period $t + 1$ is given by:

$$(11) \quad p_{it+1}^* = \phi m_{t+1} + (1 - \phi)p_{t+1}$$

From equation (16) in Problem 6.7, the aggregate price level in an odd period is:

$$(12) \quad p_{t+1} = E_{t-1} m_{t+1} + \frac{\phi(1 - f)}{1 - (1 - \phi)(1 - f)} (E_t m_{t+1} - E_{t-1} m_{t+1})$$

Since $E_{t-1} m_{t-1} = m_{t-1}$ and $E_t m_{t+1} = m_t$, we have:

$$(13) \quad p_{t+1} = m_{t-1} + \frac{\phi(1-f)}{1-(1-\phi)(1-f)} (m_t - m_{t-1})$$

Substituting equation (13) into equation (11) yields the optimal price in period $t+1$:

$$(14) \quad p_{it+1}^* = \phi m_{t+1} + (1-\phi)m_{t-1} + \frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} (m_t - m_{t-1})$$

Thus the amount of profit a firm expects to lose in period $t+1$ is proportional to:

$$(15) \quad E_t (p_{it+1} - p_{it+1}^*)^2 = E_t \left[m_{t-1} - \phi m_{t+1} - (1-\phi)m_{t-1} - \frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} (m_t - m_{t-1}) \right]^2$$

Collecting terms gives us:

$$(16) \quad E_t (p_{it+1} - p_{it+1}^*)^2 = E_t \left[\phi(m_{t-1} - m_{t+1}) + \frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} (m_{t-1} - m_t) \right]^2$$

Expanding the right-hand side of equation (16) yields:

$$(17) \quad E_t (p_{it+1} - p_{it+1}^*)^2 = \phi^2 E_t (m_{t-1} - m_{t+1})^2 + \frac{2\phi(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} E_t (m_{t-1} - m_{t+1})(m_{t-1} - m_t) + \left[\frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} \right]^2 E_t (m_{t-1} - m_t)^2$$

Note that since m follows a random walk, we have:

$$E_t (m_{t-1} - m_{t+1})(m_{t-1} - m_t) = E_t (m_{t-1} - m_{t+1}) E_t (m_{t-1} - m_t) = (m_{t-1} - m_t)(m_{t-1} - m_{t+1}) = 0$$

Thus the second term on the right-hand side of equation (17) is equal to 0. Using this fact, we can add equations (8) and (17). Thus the total amount of profit a firm setting prices in odd periods expects to lose, $E_t (p_{it} - p_{it}^*)^2 + E_t (p_{it+1} - p_{it+1}^*)^2$, is proportional to:

$$(18) \quad \text{Exp. Loss}_{\text{odd}} = \phi^2 E_t (m_{t-1} - m_t)^2 + \phi^2 E_t (m_{t-1} - m_{t+1})^2 + \left[\frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} \right]^2 E_t (m_{t-1} - m_t)^2$$

Now we will deal with firms that set price in even periods. When period t is an odd period, analysis parallel to that used to derive (8) shows that the amount of profit a firm expects to lose in period t is proportional to:

$$(19) \quad E_t (p_{it} - p_{it}^*)^2 = \phi^2 E_t (m_{t-1} - m_t)^2$$

Analysis parallel to that used to derive (17) shows that the amount of profit a firm expects to lose in period $t+1$ is proportional to:

$$(20) \quad E_t (p_{it+1} - p_{it+1}^*)^2 = \phi^2 E_t (m_{t-1} - m_{t+1})^2 + \frac{2\phi(1-\phi)\phi f}{1-(1-\phi)f} E_t (m_{t-1} - m_{t+1})(m_{t-1} - m_t) + \left[\frac{(1-\phi)\phi f}{1-(1-\phi)f} \right]^2 E_t (m_{t-1} - m_t)^2$$

Note that this is identical to (17) except that the roles of f and $(1-f)$ are reversed. Proceeding as above, we note that since m follows a random walk:

$$E_t (m_{t-1} - m_{t+1})(m_{t-1} - m_t) = E_t (m_{t-1} - m_{t+1}) E_t (m_{t-1} - m_t) = (m_{t-1} - m_t)(m_{t-1} - m_{t+1}) = 0$$

Thus the second term on the right-hand side of equation (20) is equal to 0. Using this fact, we can add equations (19) and (20). Thus the total amount of profit a firm setting prices in even periods expects to lose, $E_t (p_{it} - p_{it}^*)^2 + E_t (p_{it+1} - p_{it+1}^*)^2$, is proportional to:

$$(21) \text{ Exp. Loss}_{\text{even}} = \phi^2 E_t (m_{t-1} - m_t)^2 + \phi^2 E_t (m_{t-1} - m_{t+1})^2 + \left[\frac{(1-\phi)\phi f}{1-(1-\phi)f} \right]^2 E_t (m_{t-1} - m_t)^2$$

We need to compare the right-hand sides of equations (18) and (21). Recall that f is the fraction of firms that set prices in odd periods. Note that with $f < 1/2$ — more firms setting prices in even periods than in odd periods — we have $(1-f) > f$. Using this and the fact that $\phi < 1$, we can say that:

$$(22) \left[\frac{(1-\phi)\phi(1-f)}{1-(1-\phi)(1-f)} \right]^2 > \left[\frac{(1-\phi)\phi f}{1-(1-\phi)f} \right]^2$$

since $(1-\phi)\phi(1-f) > (1-\phi)\phi f$ and $1-(1-\phi)(1-f) < 1-(1-\phi)f$.

Thus the right-hand side of equation (18) is greater than the right-hand side of equation (21). This means that the profit a firm expects to lose if it sets prices in odd periods exceeds the profit a firm expects to lose if it sets prices in even periods. Thus it is not optimal to set prices in odd periods and firms would like to switch to setting prices in even periods. This means that with $\phi < 1$, if we start with $f < 1/2$, we would expect to see f go to zero — no firms will set price in odd periods.

By reasoning analogous to that above, we could show that if $f > 1/2$, the inequality in (22) is reversed. Firms setting prices in even periods expect to lose more than firms setting prices in odd periods. Thus it is not optimal to set price in even periods and firms would like to switch to setting prices in odd periods. This means that with $\phi < 1$, if we start with $f > 1/2$, we would expect to see f go to 1 — all firms will set price in odd periods.

Thus if $\phi < 1$, staggered price setting with $f = 1/2$ is not a stable equilibrium. If the economy starts with anything other than $f = 1/2$, staggered price setting will not prevail. The economy will go to a situation where all firms set prices in the same period.

Problem 6.9

The price set by firms in period t is:

$$(1) x_t = (p_t^* + E_t p_{t+1}^*)/2 = [(\phi m_t + (1-\phi)p_t) + (\phi E_t m_{t+1} + (1-\phi)E_t p_{t+1})]/2$$

where we have used the fact that $p^* = \phi m + (1-\phi)p$. Since $p_t = (x_t + x_{t-1})/2$ and $E_t m_{t+1} = 0$, equation (1) can be rewritten as:

$$(2) x_t = \phi m_t + \frac{(1-\phi)(x_t + x_{t-1})}{2} + \frac{[(1-\phi)(x_t + E_t x_{t+1})]/2}{2}$$

which simplifies to:

$$(3) x_t = \frac{\phi m_t}{2} + \frac{(1-\phi)(x_{t-1} + 2x_t + E_t x_{t+1})}{4}$$

Solving for x_t yields:

$$(4) x_t = A(x_{t-1} + E_t x_{t+1}) + [(1-2A)/2]m_t$$

where $A \equiv \frac{1}{2} \frac{1-\phi}{1+\phi}$.

We need to eliminate $E_t x_{t+1}$ from the expression in (4). As in the text, it is reasonable to guess that x_t is a linear function of x_{t-1} and m_t , or:

$$(5) x_t = \mu + \lambda x_{t-1} + \nu m_t$$

We need to determine whether there are actually values of μ , λ and ν that solve the model. As explained in the text, the flexible-price equilibrium involves each price equaling m . In light of this, consider a situation

where x_{t-1} and m_t are both equal to zero. If period- t price-setters also set their prices to $m_t = 0$, the economy is at its flexible-price equilibrium. In addition, since m is white noise, the period- t price-setters have no reason to expect m_{t+1} to be on average either more or less than zero, and hence no reason to expect x_{t+1} to depart on average from zero. Thus in this situation, p_{it}^* and $E_t p_{it+1}^*$ are both equal to zero and so price-setters would choose $x_t = 0$. In summary, it is reasonable to guess that when $x_{t-1} = m_t = 0$, $x_t = 0$. In terms of equation (5), this condition is:

$$(6) \quad 0 = \mu + \lambda(0) + v(0)$$

or simply $\mu = 0$. Thus equation (5) becomes:

$$(7) \quad x_t = \lambda x_{t-1} + v m_t$$

Our goal is to find values of λ and v that solve the model. Since equation (7) holds each period, it implies that $x_{t+1} = \lambda x_t + v m_{t+1}$. Thus the expectation as of time t of x_{t+1} is λx_t since $E_t m_{t+1} = 0$. Using equation (7) to substitute for x_t yields:

$$(8) \quad E_t x_{t+1} = \lambda^2 x_{t-1} + \lambda v m_t$$

Substituting equation (8) into equation (4) gives us:

$$(9) \quad x_t = A(x_{t-1} + \lambda^2 x_{t-1} + \lambda v m_t) + [(1 - 2A)/2] m_t$$

which implies:

$$(10) \quad x_t = (A + A\lambda^2) x_{t-1} + \{A\lambda v + [(1 - 2A)/2]\} m_t$$

The coefficients on x_{t-1} and m_t must be the same in equations (7) and (10). This requires:

$$(11) \quad A + A\lambda^2 = \lambda \quad \text{and} \quad (12) \quad A\lambda v + (1 - 2A)/2 = \gamma$$

Equation (11) is the same as equation (6.68) in the text for the version of the model where m follows a random walk. The solution to this quadratic is thus given by:

$$(13) \quad \lambda = \frac{1 - \sqrt{\phi}}{1 + \sqrt{\phi}}$$

Solving for v in equation (12) yields:

$$(14) \quad v(1 - A\lambda) = (1 - 2A)/2$$

From equation (11), dividing through by λ and rearranging, gives us $1 - A\lambda = A\lambda$. Substituting this expression into equation (14) gives us:

$$(15) \quad v = \frac{1 - 2A}{2} \frac{\lambda}{A}$$

Substituting equation (15) into equation (7) yields:

$$(16) \quad x_t = \lambda x_{t-1} + \frac{1 - 2A}{2} \frac{\lambda}{A} m_t$$

Thus equation (16) with λ given by equation (13) solves the model.

We can now describe the behavior of output. Using the definitions of A and λ , some simple algebra allows us to rewrite equation (16) as:

$$(17) \quad x_t = \lambda x_{t-1} + \frac{2\phi}{(1 + \sqrt{\phi})^2} m_t = \lambda x_{t-1} + C m_t$$

where we have defined $C \equiv \frac{2\phi}{(1 + \sqrt{\phi})^2}$.

Since $y_t = m_t - p_t$ and $p_t = (x_t + x_{t-1})/2$, we have:

$$(18) \quad y_t = m_t - [(x_t + x_{t-1})]/2$$

Substituting equation (17) and equation (17) lagged one period, into equation (18) yields:

$$(19) y_t = m_t - [(\lambda x_{t-1} + C m_t + \lambda x_{t-2} + C m_{t-1})/2]$$

or simply:

$$(20) y_t = m_t - \lambda p_{t-1} - [(C/2)m_t] - [(C/2)m_{t-1}]$$

where we have used the fact that $p_{t-1} = (x_{t-1} + x_{t-2})/2$. Now since $y_{t-1} = m_{t-1} - p_{t-1}$, this implies:

$$(21) y_t = m_t + \lambda y_{t-1} - \lambda m_{t-1} - (C/2)m_t - (C/2)m_{t-1}$$

Collecting terms yields:

$$(22) y_t = \left(1 - \frac{C}{2}\right)m_t - \left(\lambda + \frac{C}{2}\right)m_{t-1} + \lambda y_{t-1}$$

Finally, since:

$$(23) 1 - \frac{C}{2} = 1 - \frac{\phi}{(1 + \sqrt{\phi})^2} = \frac{1 + 2\sqrt{\phi} + \phi - \phi}{(1 + \sqrt{\phi})^2} = \frac{1 + 2\sqrt{\phi}}{(1 + \sqrt{\phi})^2}$$

and:

$$(24) \lambda + \frac{C}{2} = \frac{1 - \sqrt{\phi}}{1 + \sqrt{\phi}} + \frac{\phi}{(1 + \sqrt{\phi})^2} = \frac{(1 - \sqrt{\phi})(1 + \sqrt{\phi}) + \phi}{(1 + \sqrt{\phi})^2} = \frac{1 - \phi + \phi}{(1 + \sqrt{\phi})^2} = \frac{1}{(1 + \sqrt{\phi})^2}$$

equation (22) can be rewritten as:

$$(25) y_t = \lambda y_{t-1} \frac{1 + 2\sqrt{\phi}}{(1 + \sqrt{\phi})^2} \varepsilon_t - \frac{1}{(1 + \sqrt{\phi})^2} \varepsilon_{t-1}$$

where we have substituted for $m_t = \varepsilon_t$ and $m_{t-1} = \varepsilon_{t-1}$. Thus if the money stock is white noise, output is an ARMA(1, 1) process rather than an AR(1) process.

Problem 6.10

We could proceed as in the text, and obtain equation (6.80), which holds for a general process for m :

$$(6.80) x_t = \lambda x_{t-1} + \frac{\lambda(1-2A)}{A} \left\{ m_t + (1+\lambda) \left[E_t m_{t+1} + \lambda E_t m_{t+2} + \lambda^2 E_t m_{t+3} + \dots \right] \right\}$$

When the money stock is white noise, $E_t m_{t+s} = 0$ for all $s > 0$. Thus textbook equation (6.80) simplifies to:

$$(1) x_t = \lambda x_{t-1} + \frac{\lambda(1-2A)}{A} m_t$$

Note that equation (1) is identical to equation (16) in Problem 6.9. We could then proceed, as in Problem 6.9, to determine the behavior of output.

Problem 6.11

a) The price set by individuals at time t is:

$$(1) x(t) = \frac{1}{T} \int_{\tau=0}^T E_t m(t+\tau) d\tau = \frac{1}{T} \int_{\tau=0}^T (t+\tau) g d\tau$$

where we have substituted for the fact that $m(t) = gt$ and thus that $E_t m(t+\tau) = g(t+\tau)$. Solving the integral gives us:

$$(2) \int_{\tau=0}^T (t+\tau) g d\tau = \left[g t \tau \right]_{\tau=0}^{\tau=T} + \left[\frac{1}{2} g \tau^2 \right]_{\tau=0}^{\tau=T} = g t T + \frac{1}{2} g T^2$$

Substituting equation (2) back into equation (1) gives us the price set by individuals at time t :

$$(3) x(t) = g t + (g T / 2)$$

The aggregate price level is the average of the prices set over the last interval of length T . Thus:

$$(4) p(t) = \frac{1}{T} \int_{\tau=0}^T x(t-\tau) d\tau$$

Substituting equation (3) into equation (4) yields:

$$(5) \quad p(t) = \frac{1}{T} \int_{\tau=0}^T \left[g(t-\tau) + \frac{1}{2}gT \right] d\tau$$

Solving the integral gives us:

$$(6) \quad \int_{\tau=0}^T \left[g(t-\tau) + \frac{1}{2}gT \right] d\tau = \left[g\tau - \frac{1}{2}g\tau^2 + \frac{1}{2}gT\tau \right]_{\tau=0}^{\tau=T} = gT - \frac{1}{2}gT^2 + \frac{1}{2}gT^2 = gT$$

Substituting equation (6) back into equation (5) gives us the price level at time t :

$$(7) \quad p(t) = gt$$

Substituting $m(t) = gt$ and $p(t) = gt$ into $y(t) = m(t) - p(t)$ gives us:

$$(8) \quad y(t) = 0$$

b) i) Suppose that $x(t) = gT/2$ for all $t > 0$. Then for $t > T$, since $p(t)$ is just the average of the x 's set over the last interval of length T , $p(t) = gT/2$. Now we know that for $t > T$, $m(t) = gT/2$. Thus for $t > T$, we do have $p(t) = m(t)$. From $y(t) = m(t) - p(t)$, this means that for all $t > T$, $y(t) = 0$ — which would have been its value in the absence of the change in policy.

Now consider the situation for some time t between time 0 and time T . From time 0 to time t , we are assuming that individuals set price equal to $gT/2$. From equation (3), we know that before time 0, individuals set price equal to $gt + (gT/2)$. The aggregate price at time t , which is the average of the prices set by individuals over the past interval of length T , is therefore given by:

$$(9) \quad p(t) = \frac{1}{T} \left[\int_{\tau=t-T}^0 \left(g\tau + \frac{gT}{2} \right) d\tau + \int_{\tau=0}^t \left(\frac{gT}{2} \right) d\tau \right] = \frac{1}{T} \left[\int_{\tau=t-T}^0 \left(g\tau + \frac{gT}{2} \right) d\tau + \frac{gTt}{2} \right]$$

Solving the remaining integral in equation (9):

$$(10) \quad \int_{\tau=t-T}^0 \left(g\tau + \frac{gT}{2} \right) d\tau = \left[\frac{g\tau^2}{2} + \frac{gT\tau}{2} \right]_{\tau=t-T}^{\tau=0} = -\frac{g(t-T)^2}{2} - \frac{gT(t-T)}{2}$$

Substituting equation (10) back into equation (9) and expanding yields:

$$(11) \quad p(t) = \frac{1}{T} \left[-\frac{g(t-T)^2}{2} - \frac{gT(t-T)}{2} + \frac{gTt}{2} \right] = \frac{1}{T} \left[\frac{-gt^2 + 2gtT - gT^2 - gTt + gT^2 + gTt}{2} \right]$$

which implies:

$$(12) \quad p(t) = gt - \frac{gt^2}{2T} = gt \left[1 - \frac{t}{2T} \right]$$

Thus $p(t) = m(t)$ for t between 0 and T as well. Thus if $x(t) = gT/2$ for all $t > 0$, then $p(t) = m(t)$ for all $t > 0$, and thus output is the same as it would be without the change in policy.

b) ii) At time t , individuals set their prices equal to the average of the expected money supply over the next interval of length T . We know that $m(t) = gT/2$ for $t \geq T$, but it is strictly less than $gT/2$ for $t < T$. Thus individuals setting prices at some time t before T , are going to set their prices less than $gT/2$. Why? They will be averaging some m 's equal to $gT/2$ with some m 's less than $gT/2$ and so we must have $x(t) < gT/2$ for $0 < t < T$. For $T \leq t < 2T$, the money supply is expected to be constant at $gT/2$ and thus individuals set prices equal to this constant money supply. Thus $x(t) = gT/2$ for $T \leq t < 2T$.

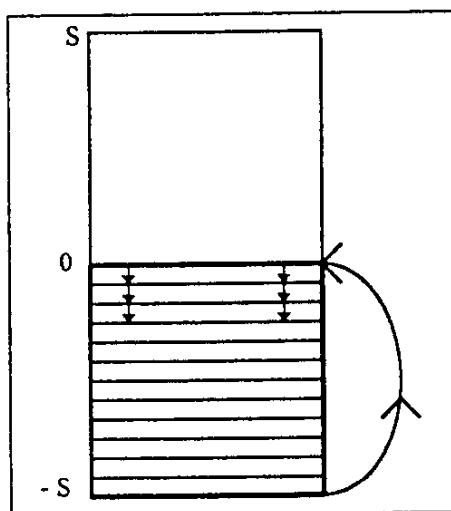
We have shown in part (b), (i), that if everyone sets prices to $gT/2$, $p(t) = m(t)$ and thus $y(t) = 0$, which is its value in the absence of the change in policy. But as we have just explained, individuals actually set prices less than $gT/2$ for $0 < t < T$. Thus the aggregate price level will be less than $m(t)$ over the interval $0 < t < 2T$. Since $y(t) = m(t) - p(t)$, this means that output will be greater than zero during this interval. Thus this steady reduction in money growth actually causes output to be higher than it would have been in the absence of the policy change.

Problem 6.12

a) Suppose first that the elevator is not at the top or bottom of the shaft. Now assume that the money supply rises by a small (formally, infinitesimal) amount dm . Since $p_i - p_i^*$ does not equal S or $-S$ for anyone, no prices change. All the $(p_i - p_i^*)$'s fall by dm . The elevator moves down the shaft by dm and stays of height S . Similarly, if m falls, no prices change. The elevator moves up the shaft by dm and stays of height S .

Now suppose the elevator is at the bottom of the shaft. Assume that the money supply rises by dm . Firms that initially have $p_i - p_i^*$ "just above" $-S$ reach the barrier. They therefore raise their price so that $p_i - p_i^* = 0$. Everyone else moves down the shaft by dm . Since the height of the elevator was S , the top of the elevator was initially at zero.

In the diagram at right, the horizontal lines represent "slices" of the elevator of infinitesimal height dm . Essentially, the firms at the bottom of the elevator jump up to the top and everyone else moves down by dm . So the elevator does not move or change shape. Thus, with an infinitesimal change in m , the distribution of $p_i - p_i^*$ is unchanged, just as in the Caplin-Spulber model.



The situation where the elevator is at the bottom of the shaft and m falls is similar to the case where the elevator is not at the top or bottom of the shaft. It simply moves up by dm .

Finally, the case where the elevator is at the top of the shaft is the reverse of the case where it starts at the bottom. If m falls, the elevator does not move. If m rises, the elevator moves down by dm .

b) For an increase in m , average price is unchanged except if the elevator is at the bottom of the shaft. In this case, the average price rises exactly as much as m . Thus, on average, increases in the money supply increase output.

Problem 6.13

a) Substitute the aggregate price level $-p = qp' + (1 - q)p^f$ into the expression for the price set by flexible-price firms $-p^f = (1 - \phi)p + \phi m$ -- to yield:

$$(1) \quad p^f = (1 - \phi)[qp' + (1 - q)p^f] + \phi m$$

Solving for p^f yields:

$$(2) \quad p^f [1 - (1 - \phi)(1 - q)] = (1 - \phi)qp' + \phi m$$

Since $1 - (1 - \phi)(1 - q) = q + \phi - \phi q = \phi + (1 - \phi)q$, equation (2) can be rewritten as:

$$(3) \quad p^f [\phi + (1 - \phi)q] = (1 - \phi)qp' + \phi m$$

and thus finally:

$$(4) \quad p^f = \frac{(1-\phi)q}{\phi + (1-\phi)q} p^r + \frac{\phi}{\phi + (1-\phi)q} m = p^r + \frac{\phi}{\phi + (1-\phi)q} (m - p^r)$$

b) Since rigid-price firms set $p^r = (1-\phi)E_p + \phi E_m$, we need to solve for E_p — the expectation of the aggregate price level. Taking the expected value of both sides of $p = qp^r + (1-q)p^f$ gives us:

$$(5) \quad E_p = qp^r + (1-q)E_p^f$$

Thus we have:

$$(6) \quad p^r = (1-\phi)[qp^r + (1-q)E_p^f] + \phi E_m$$

The rigid-price firms know how the flexible-price firms will set their price. That is, they know that flexible-price firms will use equation (4) to set their prices. Thus the rational expectation of the price set by the flexible-price firms is:

$$(7) \quad E_p^f = p^r + \frac{\phi}{\phi + (1-\phi)q} (E_m - p^r)$$

Substituting equation (7) into equation (6) yields:

$$(8) \quad p^r = (1-\phi) \left\{ qp^r + (1-q) \left[p^r + \frac{\phi}{\phi + (1-\phi)q} (E_m - p^r) \right] \right\} + \phi E_m$$

which implies:

$$(9) \quad p^r = (1-\phi)p^r + \phi E_m + \frac{(1-\phi)(1-q)\phi}{\phi + (1-\phi)q} (E_m - p^r)$$

Defining $C \equiv [(1-\phi)(1-q)\phi]/[\phi + (1-\phi)q]$, we can rewrite equation (9) as:

$$(10) \quad p^r [1 - (1-\phi) + C] = (\phi + C)E_m$$

or:

$$(11) \quad p^r (\phi + C) = (\phi + C)E_m$$

and thus finally:

$$(12) \quad p^r = E_m$$

Rigid price firms simply set their price equal to the expected value of the nominal money stock.

c) The aggregate price level is given by:

$$(13) \quad p = qp^r + (1-q)p^f$$

Substituting equation (4) for p^f into equation (13) yields:

$$(14) \quad p = qp^r + (1-q) \left[p^r + \frac{\phi}{\phi + (1-\phi)q} (m - p^r) \right] = p^r + \frac{(1-q)\phi}{\phi + (1-\phi)q} (m - p^r)$$

Finally, from equation (12), we know that $p^r = E_m$. Thus the aggregate price level is:

$$(15) \quad p = E_m + \frac{(1-q)\phi}{\phi + (1-\phi)q} (m - E_m)$$

We know that $y = m - p$. Adding and subtracting E_m to the right-hand side of this expression yields:

$$(16) \quad y = E_m + (m - E_m) - p$$

Substituting equation (15) into equation (16) yields:

$$(17) \quad y = (m - E_m) - \frac{(1-q)\phi}{\phi + (1-\phi)q} (m - E_m) = \frac{\phi + (1-\phi)q - (1-q)\phi}{\phi + (1-\phi)q} (m - E_m)$$

which simplifies to:

$$(18) \quad y = \frac{q}{\phi + (1-\phi)q} (m - E_m)$$

i) From equations (15) and (18), we can see that anticipated changes in m affect only prices. Specifically, consider the effects of an upward shift in the entire distribution of m , with the realization of $m - E_m$ held fixed. From equation (18) we can see that this will have no effects on real output. In this case, rigid-price firms get to set their price knowing that m has changed and thus incorporate it into their price setting decision.

ii) Unanticipated changes in m affect real output. That is, a higher value of m given its distribution — that is, given E_m — does raise y as we can see from equation (18). In this case, the rigid-price firms do not get to observe the higher realization of m and cannot incorporate it into their price setting decision and hence the economy does not achieve the flexible-price equilibrium.

In addition, flexible-price firms are reluctant to allow their real prices to change. One can show that:

$$\frac{\partial y}{\partial \phi} = \frac{-(1-q)q}{[\phi + (1-\phi)q]^2} [m - E_m] < 0 \text{ for } m > E_m$$

Thus a lower value of ϕ — that is, a higher degree of "real rigidity" — leads to a higher level of output for any given positive realization of $m - E_m$. This means that the impact on real output of an unanticipated increase in aggregate demand is larger the larger is the degree of real rigidity — the more reluctant are flexible-price firms to allow their real prices to vary.

Problem 6.14

a) $\pi(y_1, r^*(y_1))$ is the profit a firm receives at aggregate output level y_1 , if it charges the profit-maximizing real price, $r^*(y_1)$. $\pi(y_1, r^*(y_0))$ is the profit a firm receives at aggregate output level y_1 if it continues to charge a real price of $r^*(y_0)$ — which was the optimal price to charge when aggregate output was y_0 . Thus $G = \pi(y_1, r^*(y_1)) - \pi(y_1, r^*(y_0))$ is the additional profit a firm would receive if, when aggregate output changes from y_0 to y_1 , the firm changes its price to its new profit-maximizing level. This represents, therefore, the firm's incentive to change its price in the face of a change in aggregate real output.

b) The second-order Taylor approximation will be of the form:

$$(1) \quad G \cong G|_{y_1=y_0} + \frac{\partial G}{\partial y_1} \Big|_{y_1=y_0} \cdot [y_1 - y_0] - \frac{1}{2} \cdot \frac{\partial^2 G}{\partial y_1^2} \Big|_{y_1=y_0} \cdot [y_1 - y_0]^2$$

Clearly, G evaluated at $y_1 = y_0$ is equal to zero. In addition:

$$(2) \quad \partial G / \partial y_1 = \pi_1(y_1, r^*(y_1)) + \pi_2(y_1, r^*(y_1)) [r^*(y_1)] - \pi_1(y_1, r^*(y_0))$$

Evaluating this derivative at $y_1 = y_0$ gives us:

$$(3) \quad \partial G / \partial y_1 |_{y_1=y_0} = \pi_1(y_0, r^*(y_0)) + \pi_2(y_0, r^*(y_0)) [r^*(y_0)] - \pi_1(y_0, r^*(y_0)) = 0$$

Since $r^*(y_0)$ is defined implicitly by $\pi_2(y_0, r^*(y_0)) = 0$, the right-hand side of equation (3) is equal to zero.

Using equation (2) to find the second derivative of G with respect to y_1 gives us:

$$(4) \quad \partial^2 G / \partial y_1^2 = \pi_{11}(y_1, r^*(y_1)) + \pi_{12}(y_1, r^*(y_1)) [r^*(y_1)] + [\pi_{21}(y_1, r^*(y_1)) + \pi_{22}(y_1, r^*(y_1)) r^*(y_1)] [r^*(y_1)] + \pi_2(y_1, r^*(y_1)) [r^{*''}(y_1)] - \pi_{11}(y_1, r^*(y_0))$$

Using the fact that $\pi_2(y_1, r^*(y_1)) = 0$ and $\pi_{12}(y_1, r^*(y_1)) = \pi_{21}(y_1, r^*(y_1))$, equation (4) becomes:

$$(5) \quad \partial^2 G / \partial y_1^2 = \pi_{11}(y_1, r^*(y_1)) + 2\pi_{12}(y_1, r^*(y_1)) [r^*(y_1)] + \pi_{22}(y_1, r^*(y_1)) [r^*(y_1)]^2 - \pi_{11}(y_1, r^*(y_0))$$

Evaluating this derivative at $y_1 = y_0$ leaves us with:

$$(6) \quad \partial^2 G / \partial y_1^2 |_{y_1=y_0} = 2\pi_{12}(y_0, r^*(y_0)) [r^*(y_0)] + \pi_{22}(y_0, r^*(y_0)) [r^*(y_0)]^2$$

Now differentiate both sides of the equation that implicitly defines $r^*(y_0) - \pi_2(y_0, r^*(y_0)) = 0$ — with respect to y_0 :

$$(7) \pi_{21}(y_0, r^*(y_0)) + \pi_{22}(y_0, r^*(y_0))[r^*(y_0)]' = 0$$

and thus:

$$(8) \pi_{21}(y_0, r^*(y_0)) = -\pi_{22}(y_0, r^*(y_0))[r^*(y_0)]'$$

Substituting equation (8) into equation (6) yields:

$$(9) \partial^2 G / \partial y_1^2|_{y_1=y_0} = -2\pi_{22}(y_0, r^*(y_0))[r^*(y_0)]'^2 + \pi_{22}(y_0, r^*(y_0))[r^*(y_0)]'^2 = -\pi_{22}(y_0, r^*(y_0))[r^*(y_0)]'^2$$

Thus, since G and $\partial G / \partial y_1$ evaluated at $y_1 = y_0$ are both equal to zero, substituting equation (9) into equation (1) gives us the second-order Taylor approximation:

$$(10) G \approx -\pi_{22}(y_0, r^*(y_0))[r^*(y_0)]'^2 [y_1 - y_0]^2 / 2$$

c) The $[r^*(y_0)]'^2$ component reflects the degree of real rigidity. It tells us how much the firm's profit-maximizing real price responds to changes in aggregate real output. The $\pi_{22}(y_0, r^*(y_0))$ component reflects insensitivity of the profit function. It tells us the curvature of the profit function and thus the cost in lost profits from the firm allowing its real price to differ from its profit-maximizing value.

Problem 6.15

a) Substituting the expression for aggregate demand — $y = m - p$ — into the equation that defines the optimal price for firms — $p^* = p + \phi y$ — yields $p^* = p + \phi(m - p)$ or simply:

$$(1) p^* = (1 - \phi)p + \phi m$$

Substituting the aggregate price level — $p = fp^*$ — and $m = m'$ into equation (1) yields

$$p^* = (1 - \phi)fp^* + \phi m'$$

Solving for p^* gives us:

$$(2) p^* = \frac{\phi}{1 - (1 - \phi)f} m'$$

Now substitute equation (2) into the expression for the aggregate price level — $p = fp^*$ — to obtain:

$$(3) p = \frac{\phi f}{1 - (1 - \phi)f} m'$$

Substituting equation (3) and $m = m'$ into the expression for aggregate demand — $y = m - p$ — yields:

$$y = m' - \frac{\phi f}{1 - (1 - \phi)f} m' = \left[\frac{1 - f + \phi f - \phi f}{1 - (1 - \phi)f} \right] m'$$

or simply:

$$(4) y = \frac{(1 - f)}{1 - (1 - \phi)f} m'$$

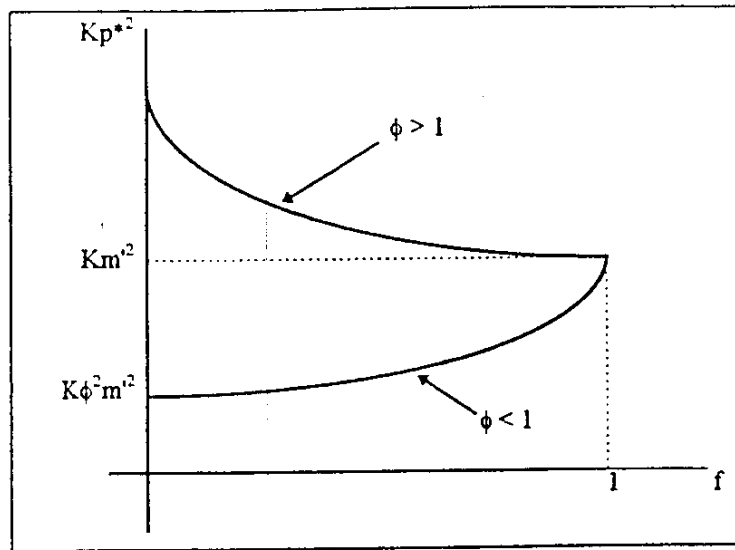
b) Substituting equation (2) — the expression for a firm's optimal price — into the expression describing the firm's incentive to adjust its price, Kp^{*2} , yields:

$$(5) Kp^{*2} = K \left[\frac{\phi m'}{1 - (1 - \phi)f} \right]^2$$

We need to plot this incentive to change price as a function of f , the fraction of firms that change their price. The following derivatives will be useful:

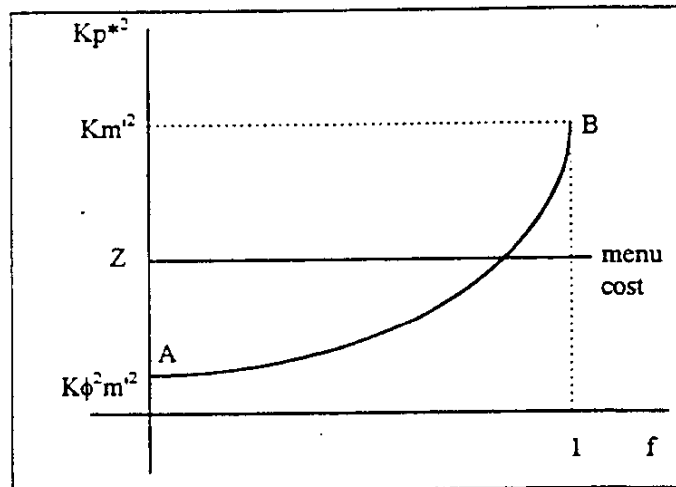
$$(6) \frac{\partial [Kp^{*2}]}{\partial f} = \frac{2K(1 - \phi)(\phi m')^2}{[1 - (1 - \phi)f]^3} \quad \text{and} \quad (7) \frac{\partial^2 [Kp^{*2}]}{\partial f^2} = \frac{6K(1 - \phi)^2 (\phi m')^2}{[1 - (1 - \phi)f]^4}$$

When $\phi < 1$, $\partial[Kp^{*2}]/\partial f > 0$ and $\partial^2[Kp^{*2}]/\partial f^2 > 0$. From equation (5), at $f = 1$, $Kp^{*2} = K[\phi m']^2/\phi^2 = Km'^2$. At $f = 0$, $Kp^{*2} = K\phi^2 m'^2 < Km'^2$ when $\phi < 1$. Thus when $\phi < 1$, the incentive for a firm to adjust its price is an increasing function of how many other firms change their price. See the diagram at right.



When $\phi > 1$, $\partial[Kp^{*2}]/\partial f < 0$ and $\partial^2[Kp^{*2}]/\partial f^2 > 0$. From equation (5), at $f = 1$, $Kp^{*2} = K[\phi m']^2/\phi^2 = Km'^2$. At $f = 0$, $Kp^{*2} = K\phi^2 m'^2 > Km'^2$ when $\phi > 1$. Thus when $\phi > 1$, the incentive for a firm to adjust its price is a decreasing function of how many other firms change their price. See the diagram.

c) In the case of $\phi < 1$, there can be a situation where both adjustment by all firms and adjustment by no firms are equilibria. See the diagram at right where the menu cost, Z , is assumed to be such that $K\phi^2 m'^2 < Z < Km'^2$.

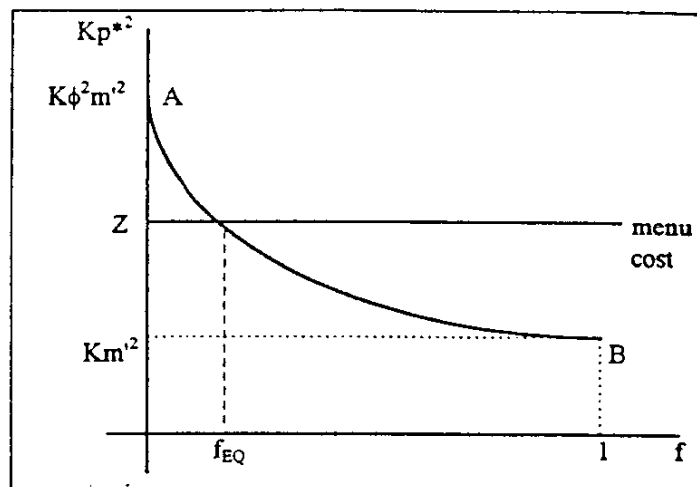


Point A is an equilibrium with $f = 0$. Consider the situation of a representative firm at point A. This says that if no one else is changing their price, the profits a firm loses by not changing its price — $Kp^{*2} = K\phi^2 m'^2$ — are less than the menu cost of Z . Thus it is optimal for the representative firm to not change its price. This is true for all firms and thus no one changing price is an equilibrium.

Point B is also an equilibrium with $f = 1$. Consider the situation of a representative firm at point B. This says that if everyone else is changing their price, the profits a firm loses by not changing its price — $Kp^{*2} = Km'^2$ — exceed the menu cost of Z . Thus it is optimal for the representative firm to change its price. This is true for all firms and thus everyone changing price is also an equilibrium.

In the case of $\phi > 1$, there can be a situation where neither adjustment by all firms nor adjustment by no firms are equilibria. See the diagram at right where the menu cost, Z , is assumed to be such that $Km'^2 < Z < K\phi^2m'^2$.

Consider the situation of $f = 0$ at point A. This says that if no one else is changing their price, the profits that a representative firm would lose by not changing price — $Kp^{*2} = K\phi^2m'^2$ — exceed the menu cost Z . Thus it is optimal for the firm to change its price. This is true for all firms and thus it cannot be an equilibrium for no one to change their price.



Now consider the situation of $f = 1$ at point B. This says that if everyone else is changing their price, the profits that a representative firm would lose by not changing their price — $Kp^{*2} = Km'^2$ — are less than the menu cost of Z . Thus it is optimal for the representative firm to not change its price. This is true for all firms and thus it cannot be an equilibrium for all firms to change their price.

From this discussion, we can see that the equilibrium in this case is for fraction f_{EQ} of firms to change their prices. If fraction f_{EQ} of firms are changing their prices, the profit that a representative firm would lose by not changing its price is exactly equal to the menu cost, Z . Thus the representative firm is indifferent and there is no tendency for the economy to move away from this point where fraction f_{EQ} of firms changing their prices.

Problem 6.16

a) We can use the intuitive reasoning employed to explain equation (10.29) in Chapter 10. Consider an asset that "pays" $-c$ when the individual climbs a palm tree and pays $\bar{u}$ when an individual trades and eats another's coconut. Assume that this asset is being priced by risk-neutral investors with required rate of return equal to r — the individual's discount rate. Since the expected present value of this asset is the same as the individual's expected value of lifetime utility, the asset must have price V_P while the individual is looking for palm trees and price V_C while the individual is looking for other people with coconuts.

For the asset to be held, it must provide an expected rate of return of r . That is, its dividends per unit time plus any expected capital gains or losses per unit time, must equal rV_P . When the individual is looking for palm trees, there are no dividends per unit time. There is a probability b per unit time of a capital "gain" of $(V_C - V_P) - c$ — if the individual finds a palm tree and climbs it, the difference in the price of the asset is $V_C - V_P$ and the asset "pays" $-c$ at that time. Thus we have:

$$(1) \quad rV_P = b(V_C - V_P - c)$$

b) The asset must have price V_C while the individual is looking for others with coconuts and must provide an expected rate of return of r . Thus its dividends per unit time plus any expected capital gains or losses per unit time, must equal rV_C . When the individual is looking for others with coconuts, there are no dividends per unit time. There is a probability aL per unit time of a capital gain of $(V_P - V_C) + \bar{u}$ — if the individual finds someone else with a coconut, trades and eats that coconut, the change in the price of the asset is $(V_P - V_C)$ and the asset pays $\bar{u}$ at that time. Thus we have:

SOLUTIONS TO CHAPTER 7

Problem 7.1

Since transitory income is on average equal to zero, we can interpret average income as average permanent income. Thus we are told that on average, farmers have lower permanent income than nonfarmers do, or $\bar{Y}_F^P < \bar{Y}_{NF}^P$. We can interpret the fact that farmers' incomes fluctuate more from year to year as meaning that the variance of transitory income for farmers is larger than the variance of transitory income for nonfarmers, or $\text{var}(Y_F^T) > \text{var}(Y_{NF}^T)$.

Consider the regression:

$$(1) C_i = a + bY_i + e_i$$

where C_i is current consumption -- which according to the permanent-income hypothesis is determined entirely by Y^P so that $C = Y^P$ -- and Y_i is current income -- which is assumed to be the sum of permanent income and transitory income so that $Y = Y^P + Y^T$. From equation (7.8) in the text, the OLS estimator of b takes the form:

$$(2) \hat{b} = \frac{\text{var}(Y^P)}{\text{var}(Y^P) + \text{var}(Y^T)}$$

As long as $\text{var}(Y^P)$ is the same across the two groups, the fact that $\text{var}(Y_F^T) > \text{var}(Y_{NF}^T)$ means that the estimated slope coefficient should be smaller for farmers than it is for nonfarmers. This means that the estimated impact on consumption of a marginal increase in current income is smaller for farmers than for nonfarmers. According to the permanent-income hypothesis, this is because the increase is much more likely to be due to transitory income for farmers than for nonfarmers. Thus it can be expected to have a smaller impact on consumption for farmers than for nonfarmers.

From equation (7.9) in the text, the OLS estimator of the constant term takes the form:

$$(3) \hat{a} = (1 - \hat{b})\bar{Y}^P$$

The fact that farmers, on average, have lower permanent incomes than nonfarmers tends to make the estimated constant term smaller for farmers. However, as was just explained, $\hat{b}$ is smaller for farmers than it is for nonfarmers. This tends to make the estimated constant term bigger for farmers than for nonfarmers. Thus the effect on the estimated constant term is ambiguous.

We can, say, however, that at the average level of permanent income for farmers, the estimated consumption function for farmers is expected to lie below the one for nonfarmers. Thus if the two estimated consumption functions do cross, they cross at a level of income less than $\bar{Y}_F^P$. Why? Consider a member of each group whose income equals the average income among farmers. Since there are many more nonfarmers with permanent incomes above this level than there are with permanent incomes below it, the nonfarmer's permanent income is much more likely to be greater than her current income than less. As a result, nonfarmers with this current income have on average higher permanent income; thus on average they consume more than their income. For the farmer, in contrast, her permanent income is about as likely to be more than current income as it is to be less; as a result, farmers with this current income on average have the same permanent income, and thus on average they consume their income. Thus the consumption function for farmers is expected to lie below the one for nonfarmers at the average level of income for farmers.

Problem 7.2

a) We need to find an expression for $[(C_{t+2} + C_{t+3})/2] - [(C_t + C_{t+1})/2]$. We can write C_{t+1} , C_{t+2} and C_{t+3} in terms of C_t and the e 's. Specifically:

$$(1) C_{t+1} = C_t + e_{t+1}$$

$$(2) C_{t+2} = C_{t+1} + e_{t+2} = C_t + e_{t+1} + e_{t+2}$$

$$(3) C_{t+3} = C_{t+2} + e_{t+3} = C_t + e_{t+1} + e_{t+2} + e_{t+3}$$

where we have used equation (1) in deriving (2) and equation (2) in deriving (3). Using equations (1) - (3), the change in measured consumption from one two-period interval to the next is:

$$(4) \frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} = \frac{(C_t + e_{t+1} + e_{t+2}) + (C_t + e_{t+1} + e_{t+2} + e_{t+3})}{2} - \frac{C_t + (C_t + e_{t+1})}{2}$$

which simplifies to:

$$(5) \frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} = \frac{e_{t+3} + 2e_{t+2} + e_{t+1}}{2}$$

b) Through similar manipulations as in part (a), the previous value of the change in measured consumption would be:

$$(6) \frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2} = \frac{e_{t+1} + 2e_t + e_{t-1}}{2}$$

Using equations (5) and (6), the covariance between successive changes in measured consumption is:

$$(7) \text{cov} \left[\left(\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} \right), \left(\frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2} \right) \right] = \text{cov} \left[\left(\frac{e_{t+3} + 2e_{t+2} + e_{t+1}}{2} \right), \left(\frac{e_{t+1} + 2e_t + e_{t-1}}{2} \right) \right]$$

Since the e 's are uncorrelated with each other and since e_{t+1} is the only value of e that appears in both expressions, the covariance reduces to:

$$(8) \text{cov} \left[\left(\frac{C_{t+2} + C_{t+3}}{2} - \frac{C_t + C_{t+1}}{2} \right), \left(\frac{C_t + C_{t+1}}{2} - \frac{C_{t-2} + C_{t-1}}{2} \right) \right] = \frac{\sigma_e^2}{4}$$

where σ_e^2 denotes the variance of the e 's. So the change in measured consumption is correlated with its previous value. Since the covariance is positive, this means that if measured consumption in the two-period interval $(t, t+1)$ is greater than measured consumption in the two-period interval $(t-2, t-1)$, measured consumption in $(t+2, t+3)$ will tend to be greater than measured consumption in $(t, t+1)$. When a variable follows a random walk, successive changes in the variable are uncorrelated. For example, with actual consumption in this model, we have $C_t - C_{t-1} = e_t$ while $C_{t+1} - C_t = e_{t+1}$. Since e_t and e_{t+1} are uncorrelated, successive changes in actual consumption are uncorrelated. Thus if C_t is bigger than C_{t-1} this does not mean that C_{t+1} will tend to be above C_t . Since successive changes in measured consumption are correlated, measured consumption is not a random walk. The change in measured consumption today does provide us with some information as to what the change in measured consumption is likely to be tomorrow.

c) From equation (5), the change in measured consumption from $(t, t+1)$ to $(t+2, t+3)$ depends on e_{t+1} , the innovation to consumption in period $t+1$. But this is known as of $t+1$, which is part of the first two-period interval. Thus the change in consumption from one two-period interval to the next is not uncorrelated with everything known as of the first two-period interval. However, it is uncorrelated with everything known in the two-period interval immediately preceding $(t, t+1)$. From equation (5), e_{t+3} , e_{t+2} and e_{t+1} are all unknown as of the two-period interval $(t-2, t-1)$.

d) We can write C_{t+3} as a function of C_{t+1} and the e 's. Specifically:

$$(9) C_{t+3} = C_{t+2} + e_{t+3} = C_{t+1} + e_{t+2} + e_{t+3}$$

Thus the change in measured consumption from one two-period interval to the next is:

$$(10) C_{t+3} - C_{t+1} = C_{t+1} + e_{t+2} + e_{t+3} - C_{t+1} = e_{t+2} + e_{t+3}$$

The same calculations would yield the previous value of the change in measured consumption:

$$(11) C_{t+1} - C_{t-1} = e_t + e_{t+1}$$

And so the covariance between successive changes in measured consumption is:

$$(12) \text{cov}[(C_{t+3} - C_{t+1}), (C_{t+1} - C_{t-1})] = \text{cov}[(e_{t+2} + e_{t+3}), (e_t + e_{t+1})]$$

Since the e 's are uncorrelated with each other, the covariance is zero. Thus measured consumption is a random walk in this case. The amount that C_{t+1} differs from C_{t-1} does not provide any information as to what the difference between C_{t+1} and C_{t+3} will be.

Problem 7.3

a) Consider the usual experiment of a decrease in consumption by a small (formally, infinitesimal) amount dC in period t . With the CRRA utility function given by:

$$(1) u(C_t) = C_t^{1-\theta} / (1-\theta)$$

the marginal utility of consumption in period t is $C_t^{-\theta}$. Thus the change has a utility cost of:

$$(2) \text{utility cost} = C_t^{-\theta} dC$$

The marginal utility of consumption in period $t+1$ is $C_{t+1}^{-\theta}$. With a real interest rate of r , the individual gets to consume an additional $(1+r)dC$ in period $t+1$. This has a discounted expected utility benefit of:

$$(3) \text{expected utility benefit} = \frac{1}{1+\rho} E_t [C_{t+1}^{-\theta} (1+r)dC]$$

If the individual is optimizing, a marginal change of this type does not affect expected utility. This means that the utility cost must equal the expected utility benefit or:

$$(4) C_t^{-\theta} = \frac{1+r}{1+\rho} E_t [C_{t+1}^{-\theta}]$$

where we have (rather informally) canceled the dC 's. Equation (4) is the Euler equation.

b) For any variable x , $e^{\ln x} = x$, and so we can write:

$$(5) E_t [C_{t+1}^{-\theta}] = E_t [e^{-\theta \ln C_{t+1}}]$$

Using the hint in the question -- if $x \sim N(\mu, V)$ then $E[e^x] = e^\mu e^{V/2}$ -- then since the log of consumption is distributed normally, we have:

$$(6) E_t [C_{t+1}^{-\theta}] = E_t \left[e^{-\theta E_t \ln C_{t+1}} e^{\theta^2 \sigma^2} \right]$$

$$= e^{-\theta E_t \ln C_{t+1}} e^{\theta^2 \sigma^2}$$

In the first line, we have used the fact that conditional on time t information, the variance of log consumption is σ^2 . In addition, we have written the mean of log consumption in period $t+1$, conditional on time t information, as $E_t \ln C_{t+1}$. Finally, in the last line we have used the fact that $e^{-\theta E_t \ln C_{t+1}} e^{\theta^2 \sigma^2}$ is simply a constant.

Substituting equation (6) back into equation (4) and taking the log of both sides yields:

$$(7) -\theta \ln C_t = \ln(1+r) - \ln(1+\rho) - \theta E_t \ln C_{t+1} + \theta^2 \sigma^2$$

Dividing both sides of equation (7) by $-\theta$ leaves us with:

$$(8) \ln C_t = E_t \ln C_{t+1} + [\ln(1+\rho) - \ln(1+r)]/\theta - \theta \sigma^2$$

c) Rearranging equation (8) to solve for $E_t \ln C_{t+1}$ gives us:

$$(9) E_t \ln C_{t+1} = \ln C_t + [\ln(1+r) - \ln(1+\rho)]/\theta + \theta \sigma^2$$

Equation (9) implies that consumption is expected to change by the constant amount

$[\ln(1+r) - \ln(1+\rho)]/\theta + \theta \sigma^2$ from one period to the next. Changes in consumption other than this deterministic amount are unpredictable. By the definition of expectations we can write:

$$(10) \quad \ln C_{t+1} = \ln C_t + [\ln(1+r) - \ln(1-\rho)]/\theta + \theta\sigma^2 + u_{t+1}$$

where u_{t+1} has mean zero, conditional on time t information. Thus log consumption follows a random walk with drift where $[\ln(1+r) - \ln(1-\rho)]/\theta + \theta\sigma^2$ is the drift parameter.

d) From equation (9), expected consumption growth is:

$$(11) \quad E_t [\ln C_{t+1} - \ln C_t] = [\ln(1+r) - \ln(1-\rho)]/\theta + \theta\sigma^2$$

Clearly, a rise in r raises expected consumption growth. We have:

$$(12) \quad \frac{\partial E_t [\ln C_{t+1} - \ln C_t]}{\partial r} = \frac{1}{\theta} \frac{1}{(1+r)} > 0$$

Note that the smaller is θ — the bigger is the elasticity of substitution, $1/\theta$ — the more that consumption growth increases due to a given increase in the real interest rate.

An increase in σ^2 also increases consumption growth:

$$(13) \quad \frac{\partial E_t [\ln C_{t+1} - \ln C_t]}{\partial \sigma^2} = \theta > 0$$

It is straightforward to verify that the CRRA utility function has a positive third derivative. From equation

$$(1), \quad u'(C_t) = C_t^{-\theta} \text{ and } u''(C_t) = -\theta C_t^{-\theta-1}. \text{ Thus:}$$

$$(14) \quad u'''(C_t) = -\theta(-\theta-1)C_t^{-\theta-2} = (\theta^2 + \theta)C_t^{-\theta-2} > 0$$

So an individual with a CRRA utility function exhibits the precautionary saving behavior explained in Section 7.6. A rise in uncertainty (as measured by σ^2 , the variance of log consumption) increases saving and thus expected consumption growth.

Problem 7.4

a) Substituting the expression for consumption in period t , which is:

$$(1) \quad C_t = \frac{r}{1+r} \left[A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s} \right]$$

into the expression for wealth in period $t+1$, which is:

$$(2) \quad A_{t+1} = (1+r)[A_t + Y_t - C_t]$$

gives us:

$$(3) \quad A_{t+1} = (1+r) \left[A_t + Y_t - \frac{r}{1+r} A_t - \frac{r}{1+r} \left(Y_t + \frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \dots \right) \right]$$

Obtaining a common denominator of $(1+r)$ and then canceling the $(1+r)$'s gives us:

$$(4) \quad A_{t+1} = A_t + Y_t - r \left(\frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \dots \right)$$

Since equation (1) holds in all periods, we can write consumption in period $t+1$ as:

$$(5) \quad C_{t+1} = \frac{r}{1+r} \left[A_{t+1} + \sum_{s=0}^{\infty} \frac{E_{t+1}[Y_{t+1+s}]}{(1+r)^s} \right]$$

Substituting equation (4) into equation (5) yields:

$$(6) \quad C_{t+1} = \frac{r}{1+r} \left[A_t + Y_t - r \left(\frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \dots \right) + \left(E_{t+1} Y_{t+1} + \frac{E_{t+1} Y_{t+2}}{1+r} + \dots \right) \right]$$

Taking the expectation, conditional on time t information, of both sides of equation (6) gives us:

$$(7) E_t C_{t+1} = \frac{r}{1+r} \left[A_t + Y_t - r \left(\frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \dots \right) + \left(E_t Y_{t+1} + \frac{E_t Y_{t+2}}{1+r} + \dots \right) \right]$$

where we have used the law of iterated projections so that for any variable x , $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$. If this did not hold, individuals would be expecting to revise their estimate either upwards or downwards and thus their original expectation could not have been rational. Collecting terms in equation (7) gives us:

$$(8) E_t C_{t+1} = \frac{r}{1+r} \left[A_t + Y_t - \left(1 - \frac{r}{1+r} \right) E_t Y_{t+1} + \left(\frac{1}{1+r} - \frac{r}{(1+r)^2} \right) E_t Y_{t+2} + \dots \right]$$

which simplifies to:

$$(9) E_t C_{t+1} = \frac{r}{1+r} \left[A_t + Y_t + \frac{E_t Y_{t+1}}{1+r} + \frac{E_t Y_{t+2}}{(1+r)^2} + \dots \right]$$

Using summation notation, and noting that $E_t Y_t = Y_t$, we have:

$$(10) E_t C_{t+1} = \frac{r}{1+r} \left[A_t + \sum_{s=0}^{\infty} \frac{E_t Y_{t+s}}{(1+r)^s} \right]$$

The right-hand sides of equations (1) and (10) are equal and thus:

$$(11) E_t C_{t+1} = C_t$$

Consumption follows a random walk — changes in consumption are unpredictable.

Since consumption follows a random walk, the best estimate of consumption in any future period is simply the value of consumption in this period. That is, for any $s \geq 0$, we can write:

$$(12) E_t C_{t+s} = C_t$$

Using equation (12), we can write the present value of the expected path of consumption as:

$$(13) \sum_{s=0}^{\infty} \frac{E_t [C_{t+s}]}{(1+r)^s} = \sum_{s=0}^{\infty} \frac{C_t}{(1+r)^s} = C_t \sum_{s=0}^{\infty} \frac{1}{(1+r)^s}$$

Since $1/(1+r) < 1$, the infinite sum converges to $1/[1 - 1/(1+r)] = (1+r)/r$ and thus:

$$(14) \sum_{s=0}^{\infty} \frac{E_t [C_{t+s}]}{(1+r)^s} = \frac{1+r}{r} C_t$$

Substituting equation (1) for C_t into the right-hand side of equation (14) yields:

$$(15) \sum_{s=0}^{\infty} \frac{E_t [C_{t+s}]}{(1+r)^s} = \frac{r}{1+r} \left(\frac{1+r}{r} \right) \left[A_t + \sum_{s=0}^{\infty} \frac{E_t [Y_{t+s}]}{(1+r)^s} \right] = A_t + \sum_{s=0}^{\infty} \frac{E_t [Y_{t+s}]}{(1+r)^s}$$

Equation (15) states that the present value of the expected path of consumption equals initial wealth plus the present value of the expected path of income.

b) Taking the expected value, as of time $t-1$, of both sides of equation (1) yields:

$$(16) E_{t-1} C_t = \frac{r}{1+r} \left[A_t + \sum_{s=0}^{\infty} \frac{E_{t-1} [Y_{t+s}]}{(1+r)^s} \right]$$

where we have used the fact that $A_t = (1+r)[A_{t-1} + Y_{t-1} - C_{t-1}]$ is not uncertain as of $t-1$. In addition, we have used the law of iterated projections so that $E_{t-1} E_t [Y_{t+s}] = E_{t-1} [Y_{t+s}]$. Subtracting equation (16) from equation (1) gives us the innovation in consumption:

$$(17) C_t - E_{t-1} C_t = \frac{r}{1+r} \left[\sum_{s=0}^{\infty} \frac{E_t [Y_{t+s}]}{(1+r)^s} - \sum_{s=0}^{\infty} \frac{E_{t-1} [Y_{t+s}]}{(1+r)^s} \right] = \frac{r}{1+r} \left[\sum_{s=0}^{\infty} \frac{E_t [Y_{t+s}] - E_{t-1} [Y_{t+s}]}{(1+r)^s} \right]$$

The innovation in consumption will be fraction $r/(1+r)$ of the present value of the change in expected lifetime income.

The next step is to determine the present value of the change in expected lifetime income. That is, we need to determine:

$$(18) \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = [Y_t - E_{t-1}Y_t] + \left[\frac{E_t Y_{t+1} - E_{t-1} Y_{t+1}}{1+r} \right] + \left[\frac{E_t Y_{t+2} - E_{t-1} Y_{t+2}}{(1+r)^2} \right] + \dots$$

In what follows, "expected to be higher" means "expected, as of period t , to be higher than it was, as of period $t-1$ ". We are told that $u_t = 1$ and thus:

$$(19) Y_t - E_{t-1} Y_t = 1$$

In period $t+1$, since $\Delta Y_{t+1} = \phi \Delta Y_t + u_{t+1}$, the change in Y_{t+1} is expected to be $\phi \Delta Y_t = \phi$ higher. Thus the level of Y_{t+1} is expected to be higher by $1 + \phi$. Thus:

$$(20) \frac{E_t Y_{t+1} - E_{t-1} Y_{t+1}}{1+r} = \frac{1+\phi}{1+r}$$

In period $t+2$, since $\Delta Y_{t+2} = \phi \Delta Y_{t+1} + u_{t+2}$, the change in Y_{t+2} is expected to be higher by $\phi \Delta Y_{t+1} = \phi^2$. Thus the level of Y_{t+2} is expected to be higher by $1 + \phi + \phi^2$. Therefore, we have:

$$(21) \frac{E_t Y_{t+2} - E_{t-1} Y_{t+2}}{(1+r)^2} = \frac{1+\phi+\phi^2}{(1+r)^2}$$

The pattern should be clear. We have:

$$(22) \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = 1 + \frac{1+\phi}{1+r} + \frac{1+\phi+\phi^2}{(1+r)^2} + \frac{1+\phi+\phi^2+\phi^3}{(1+r)^3} + \dots$$

Note that this infinite series can be rewritten as:

$$(23) \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots \right] + \left[\frac{\phi}{1+r} + \frac{\phi}{(1+r)^2} + \frac{\phi}{(1+r)^3} + \dots \right] + \left[\frac{\phi^2}{(1+r)^2} + \frac{\phi^2}{(1+r)^3} + \dots \right] + \dots$$

For ease of notation, define $\gamma \equiv 1/(1+r)$. Then the first sum on the right-hand side of (23) converges to $1/(1-\gamma)$. The second sum converges to $\phi\gamma/(1-\gamma)$. The third sum converges to $\phi^2\gamma^2/(1-\gamma)$. And so on.

Thus equation (23) can be rewritten as:

$$(24) \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \frac{1}{1-\gamma} [1 + \phi\gamma + \phi^2\gamma^2 + \dots] = \frac{1}{(1-\gamma)} \frac{1}{(1-\phi\gamma)}$$

Using the definition of γ to rewrite equation (24) yields:

$$(25) \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \frac{1}{1 - [1/(1+r)]} \frac{1}{1 - [\phi/(1+r)]} = \frac{(1+r)}{r} \frac{(1+r)}{(1+r-\phi)}$$

Substituting equation (25) into equation (24) gives us the change in consumption:

$$(26) C_t - E_{t-1}C_t = \frac{r}{(1+r)} \left[\frac{(1+r)}{r} \frac{(1+r)}{(1+r-\phi)} \right] = \frac{(1+r)}{(1+r-\phi)}$$

c) The variance of the innovation in consumption is:

$$(27) \text{ var}(C_t - E_{t-1}C_t) = \text{ var}\left[\frac{(1+r)}{(1+r-\phi)} u_t\right] = \left[\frac{(1+r)}{(1+r-\phi)}\right]^2 \text{ var}(u_t) > \text{ var}(u_t)$$

Since $(1+r)/(1+r-\phi) > 1$, the variance of the innovation in consumption is greater than the variance of the innovation in income. Intuitively, an innovation to income means that, on average, the consumer will experience further changes in income in the same direction in future periods.

It is not clear whether consumers use saving and borrowing to smooth consumption relative to income. Income is not stationary, so it is not obvious what it means to smooth it.

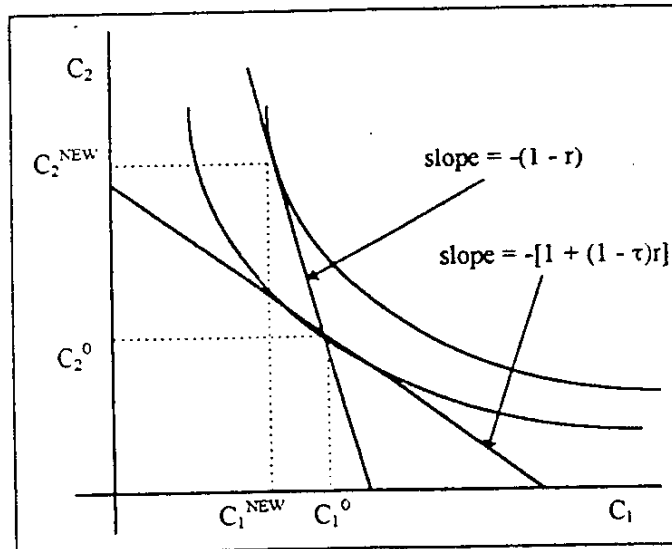
Problem 7.5

a) The present value of the lump-sum taxes is $T_1 + [T_2/(1+r)]$. The present value of the tax on interest income is $[r/(1+r)]\tau(Y_1 - C_1^0)$. The government must choose T_1 and T_2 so that these two quantities are equal, or:

$$(1) \quad T_1 + \frac{T_2}{1+r} = \frac{r}{1+r} \tau(Y_1 - C_1^0)$$

b) Suppose the new taxes satisfy condition (1). This means that at the point where the individual consumes C_1^0 , she pays the same with the new lump-sum tax as she did with the old tax on interest income. That is, right at C_1^0 , the individual's after-tax lifetime income is the same under both tax schemes. Thus at C_1^0 , the individual has just enough to consume C_2^0 in the second period under both tax schemes. This means that the new budget line must go through (C_1^0, C_2^0) just as the old one did. Since (C_1^0, C_2^0) lies right on the new budget line, it is just affordable.

c) First-period consumption must fall. Consider the diagram at right. The old budget line had slope $-[1 + (1-\tau)r]$ where τ was the tax rate on interest income. The new budget line is steeper. It has slope equal to $-(1+r)$. With saving no longer taxed, giving up one unit of period-one consumption yields more units of period-two consumption. Specifically, it yields $(1+r)$ rather than $[1 + (1-\tau)r]$.



As explained in part (b), if the government sets the new lump-sum taxes so that the present value of government revenue is unchanged, the new budget line must go through the original consumption bundle.

From the diagram, we can see that the new tangency must involve lower consumption in the first period. Intuitively, the government has set the tax rate so that there is no income effect from the change in policy, only a substitution effect. Thus, since the rate of return on saving increases, the individual chooses to save more and consume less in the first period.

Problem 7.6

a) The change in purchases in period t , dE_t , must leave the present value of spending unchanged, so that:

$$(1) \quad dE_t + dE_{t+1} + dE_{t+2} = 0$$

In addition, it must leave consumption in period $t + 2$ unchanged, or:

$$(2) (1 - \delta)^2 dE_t + (1 - \delta)dE_{t+1} + dE_{t+2} = 0$$

To see why equation (2) must hold, note that we can write the change in C_t as $dC_t = dE_t$. The change in C_{t+1} is $dC_{t+1} = (1 - \rho)dC_t + dE_{t+1}$ or substituting for dC_t , we have $dC_{t+1} = (1 - \rho)dE_t + dE_{t+1}$. The change in C_{t+2} is $dC_{t+2} = (1 - \delta)dC_{t+1} + dE_{t+2}$ or substituting for dC_{t+1} , we have $dC_{t+2} = (1 - \delta)^2 dE_t + (1 - \delta)dE_{t+1} + dE_{t+2}$. Thus if C_{t+2} is not to change, equation (2) must hold.

Thus we have two equations in two unknowns. Solving equation (1) for dE_{t+2} gives us:

$$(3) dE_{t+2} = -dE_t - dE_{t+1}$$

Substituting equation (3) into equation (2) yields:

$$(4) (1 - \delta)^2 dE_t + (1 - \delta)dE_{t+1} - dE_t - dE_{t+1} = 0$$

Expanding and collecting terms gives us:

$$(5) dE_t [1 - 2\delta + \delta^2 - 1] + [1 - \delta - 1]dE_{t+1} = 0$$

and thus:

$$(6) dE_{t+1} = (\delta - 2)dE_t$$

Substituting equation (6) into equation (3) yields:

$$(7) dE_{t+2} = -dE_t - (\delta - 2)dE_t$$

and thus:

$$(8) dE_{t+2} = (1 - \delta)dE_t$$

b) Since $C_t = (1 - \delta)C_{t-1} + E_t$, then:

$$(9) dC_t = dE_t$$

Since $C_{t+1} = (1 - \delta)C_t + E_{t+1}$, then:

$$(10) dC_{t+1} = (1 - \delta)dC_t + dE_{t+1}$$

Substituting equations (9) and (6) into equation (10) gives us:

$$(11) dC_{t+1} = (1 - \delta)dE_t + (\delta - 2)dE_t = -dE_t$$

Since only C_t and C_{t+1} are changed – C_{t+2} is unchanged by construction – we only need to look at expected utility in periods t and $t + 1$. Since instantaneous utility is quadratic, the marginal utility of consumption in period t is $1 - aC_t$. Thus the change in utility in period t is $(1 - aC_t)(dE_t)$. The marginal utility of consumption in period $t + 1$ is given by $1 - aC_{t+1}$. Since $dC_{t+1} = -dE_t$, the change in expected utility in period $t + 1$ is the expected value of $(1 - aC_{t+1})(-dE_t)$.

c) For this change in expected utility to be zero – as it must be, if the individual is optimizing – we require:

$$(12) (1 - aC_t)(dE_t) + \text{expected value of } [(1 - aC_{t+1})(-dE_t)] = 0$$

Canceling the dE_t 's (which is somewhat informal), subtracting one from both sides and then dividing both sides by $(-a)$ yields:

$$(13) \text{expected value of } C_{t+1} = C_t$$

Thus consumption follows a random walk since changes in consumption are unpredictable. The best estimate of consumption in period $t + 1$ is simply what consumption is equal to this period.

d) Rearranging $C_t = (1 - \delta)C_{t-1} + E_t$ to solve for E_t gives us:

$$(14) E_t = C_t - (1 - \delta)C_{t-1}$$

Equation (14) holds for all periods and so we can write:

$$(15) E_{t-1} = C_{t-1} - (1 - \delta)C_{t-2}$$

Subtracting equation (15) from equation (14) gives us:

$$(16) E_t - E_{t-1} = C_t - (1 - \delta)C_{t-1} - C_{t-1} + (1 - \delta)C_{t-2}$$

which implies:

$$(17) E_t - E_{t-1} = (C_t - C_{t-1}) - (1 - \delta)(C_{t-1} - C_{t-2})$$

Since consumption is a random walk, we can write:

$$(18) C_t = C_{t-1} + u_t$$

where u_t is a variable whose expectation as of $t - 1$ is zero.

Using equation (18) — and the fact that it holds in all periods — (17) can be rewritten as:

$$(19) E_t - E_{t-1} = u_t - (1 - \delta)u_{t-1}$$

Equation (19) states that the change in purchases from $t - 1$ to t has a predictable component — a component that is known as of $t - 1$ — which is u_{t-1} , the innovation to consumption in period $t - 1$. Thus purchases of durable goods will not follow a random walk.

As explained in Section 7.2, any change in expected lifetime resources is spread out equally among consumption in each remaining period of the individual's life — although we are simplifying by using a discount rate of zero, the basic ideas are general. Now suppose that in period $t - 1$, the individual's estimate of lifetime resources changes in such a way that C_{t-1} is one unit higher than C_{t-2} — that is, $u_{t-1} = 1$. This also means that expected consumption in all future periods is one unit higher than it used to be. In order to get C_{t-1} up by one, purchases in period $t - 1$ must be one higher than they were expected to be. But now look at the change in purchases from $t - 1$ to t . From equation (19), the expectation (as of period $t - 1$) of the change in purchases from $t - 1$ to t is $-(1 - \delta)$, since u_{t-1} is assumed to equal one.

Intuitively, some of the new goods purchased in period $t - 1$ will still be around in period t . Thus to keep expected consumption in period t at the new higher path — one higher than it was before — it is not expected to be necessary to buy one unit of goods all over again. The individual only has to purchase enough to replace the fraction of the extra $t - 1$ purchases that depreciated, which is fraction δ . Thus purchases in period t are expected to be less than purchases in $t - 1$. Specifically, they are expected to be lower by the amount that does not depreciate, which is $(1 - \delta)$. Thus, as of period $t - 1$, part of the change in purchases between $t - 1$ and t is predictable and thus purchases do not follow a random walk.

Now consider what happens if $\delta = 0$ — there is no depreciation. Then from equation (19), the expectation (as of period $t - 1$) of the change in purchases from $t - 1$ to t is -1 . Now all of the new goods purchased in period $t - 1$ will still be around in period t . Thus to keep expected consumption at its new higher path — one higher than it was before — it is not expected to be necessary to purchase anything new in period t . Thus purchases are expected to fall by the whole amount of the innovation in purchases the previous period.

Problem 7.7

a) Consider the following experiment. In period t , the individual reduces consumption by a small (formally, infinitesimal) amount dC and uses the proceeds to purchase stock. Since one unit of stock costs P_t , dC will buy the individual dC/P_t units of stock. This change has a utility cost of dC since utility is linear in consumption.

In period $t + 1$, the individual will receive $D_{t+1} [dC/P_t]$ in dividends which she can consume. She can then sell the stock, receiving $P_{t+1} [dC/P_t]$ which she can also consume. The discounted expected utility benefit of doing this is $E_t [1/(1 + r)][D_{t+1} + P_{t+1}][dC/P_t]$.

If the individual is optimizing, a marginal change of this type must leave expected utility unchanged. Thus the utility cost must equal the expected utility benefit, or:

$$(1) dC = E_t \left[\left(\frac{1}{1+r} \right) (D_{t+1} + P_{t+1}) \frac{dC}{P_t} \right]$$

Canceling the dC 's – which is somewhat informal – and multiplying both sides of the resulting expression by P_t yields:

$$(2) P_t = E_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right]$$

b) Equation (2) holds in all periods and so we can write:

$$(3) P_{t+1} = E_{t+1} \left[\frac{D_{t+2} + P_{t+2}}{1+r} \right]$$

Substituting equation (3) into equation (2) gives us:

$$(4) P_t = E_t \left[\frac{D_{t+1}}{1+r} \right] + E_t E_{t+1} \left[\frac{D_{t+2} + P_{t+2}}{(1+r)^2} \right]$$

Now we can use the law of iterated projections. For a variable x , $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$. Equation (4) then becomes:

$$(5) P_t = E_t \left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2} \right] + E_t \left[\frac{P_{t+2}}{(1+r)^2} \right]$$

We could now substitute for P_{t+2} and then P_{t+3} and so on. We would have:

$$(6) P_t = E_t \left[\frac{D_{t+1}}{1+r} + \frac{D_{t+2}}{(1+r)^2} + \dots + \frac{D_{t+s}}{(1+r)^s} \right] + E_t \left[\frac{P_{t+s}}{(1+r)^s} \right]$$

Imposing the no-bubbles condition that $\lim_{s \rightarrow \infty} E_t [P_{t+s} / (1+r)^s] = 0$, we can write P_t as:

$$(7) P_t = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+s}}{(1+r)^s} \right]$$

Equation (7) says that the price of the stock is the present value of the stream of expected future dividends.

Problem 7.8

a) i) With the bubble term, the price of the stock in period t is now:

$$(1) P_t = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + (1+r)^t b$$

We need to see if such a price path satisfies the individual's first-order condition, which is given by:

$$(2) P_t = E_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right]$$

Specifically, then, we need to see if the right-hand sides of equations (1) and (2) are equivalent. Since equation (1) holds every period, we can write the price of the stock in period $t+1$ as:

$$(3) P_{t+1} = \sum_{s=1}^{\infty} E_{t+1} \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + (1+r)^{t+1} b$$

Dividing both sides of equation (3) by $(1+r)$ and then taking the time t expectation of both sides of the resulting expression gives us:

$$(4) E_t \left[\frac{P_{t+1}}{1+r} \right] = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+1+s}}{(1+r)^{s+1}} \right] + (1+r)^t b$$

where we have used the law of iterated projections so that $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$ for any variable x . Now add

$E_t [D_{t+1} / (1+r)]$ to both sides of equation (4) to obtain:

$$(5) \quad E_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right] = E_t \left[\frac{D_{t+1}}{1+r} \right] + \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+1+s}}{(1+r)^{s+1}} \right] + (1+r)^t b = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + (1+r)^t b$$

Thus the right-hand sides of equations (1) and (2) are equivalent and so the proposed price path satisfies the individual's first-order condition. In this case, consumers are willing to pay more than the present value of the stream of expected future dividends. That is because they anticipate the price of the stock will keep rising so that they can enjoy capital gains that exactly offset the premium they are paying.

a) ii) If b was negative, then as $t \rightarrow \infty$, the bubble term $-(1+r)^t b$ goes to minus infinity. Thus the price of the stock eventually becomes negative and goes to minus infinity. But that is not possible. The stock will never sell for a negative price. The strategy of just holding on to the stock and never selling it would avoid the capital loss from selling at a negative price. Or even more simply, an individual could just throw her stock certificate away rather than sell it for a negative price. Thus b cannot be negative.

b) i) With this bubble term, the price of the stock in period t is:

$$(6) \quad P_t = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + q_t$$

where q_t equals $(1+r)q_{t-1}/\alpha$ with probability α and equals zero with probability $(1-\alpha)$. Again, we need to see if the right-hand side of equation (6) is equivalent to the right-hand side of equation (2), the first-order condition. Since equation (6) holds in every period, we can write the price of the stock in period $t+1$ as:

$$(7) \quad P_{t+1} = \sum_{s=1}^{\infty} E_{t+1} \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + q_{t+1}$$

Taking the time t expectation of both sides of equation (7) and using the law of iterated projections, we have:

$$(8) \quad E_t [P_{t+1}] = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + \frac{(1+r)q_t}{\alpha} \alpha + (0)(1-\alpha) = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+1+s}}{(1+r)^s} \right] + (1+r)q_t$$

Dividing both sides of equation (8) by $(1+r)$ and then adding $E_t [D_{t+1} / (1+r)]$ to both sides of the resulting expression gives us:

$$(9) \quad E_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right] = E_t \left[\frac{D_{t+1}}{1+r} \right] + \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+1+s}}{(1+r)^{s+1}} \right] + q_t = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+s}}{(1+r)^s} \right] + q_t$$

Thus the right-hand sides of equations (6) and (2) are equivalent and so the proposed price path satisfies the individual's first-order condition.

b) ii) The probability that the bubble has burst by time $t+s$ is the probability that it bursts in $t+1$ plus the probability that it bursts in $t+2$, given that it did not burst in $t+1$, plus the probability that it bursts in $t+3$, given that it did not burst in $t+1$ or $t+2$ and so on. The probability that the bubble bursts in period $t+1$ is $(1-\alpha)$. The probability that the bubble bursts in $t+2$, given that it did not burst in period $t+1$ is given by $\alpha(1-\alpha)$. The probability that the bubble bursts in $t+3$, given that it did not burst in periods $t+1$ or $t+2$ is $\alpha^2(1-\alpha)$. And so on, up to the probability that the bubble bursts in period s , given that it has not burst in any previous period, which is $\alpha^{s-1}(1-\alpha)$. Thus the probability that the bubble has burst by time $t+s$ is given by the sum of all these probabilities, or:

$$(10) \quad \text{Prob}(\text{burst by } t+s) = (1-\alpha)(1 + \alpha + \alpha^2 + \dots + \alpha^{s-1})$$

As we allow s to go to infinity, then since $\alpha < 1$, $1 + \alpha + \alpha^2 + \dots + \alpha^{s-1}$ converges to $1/(1 - \alpha)$. Thus the probability that the bubble has burst by time $t + s$, as s goes to infinity is $(1 - \alpha)/(1 - \alpha)$ or simply one.

c) i) The price of the stock in period t , in the absence of bubbles, is given by:

$$(11) \quad P_t = \sum_{s=1}^{\infty} E_t \left[\frac{D_{t+s}}{(1+r)^s} \right]$$

If dividends follow a random walk, then $E_t D_{t+s} = D_t$ for any $s \geq 0$. Since changes in dividends are unpredictable, the best estimate of dividends in any future period is what dividends are today. Thus P_t can be written as:

$$(12) \quad P_t = \sum_{s=1}^{\infty} \frac{D_t}{(1+r)^s} = D_t \sum_{s=1}^{\infty} \frac{1}{(1+r)^s} = D_t \left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots \right]$$

With $1/(1+r) < 1$, we have:

$$(13) \quad \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots = \frac{1/(1+r)}{1 - [1/(1+r)]} = \frac{1/(1+r)}{r/(1+r)} = \frac{1}{r}$$

Substituting equation (13) into equation (12) gives us the price of the stock in period t :

$$(14) \quad P_t = D_t / r$$

c) ii) With the bubble term, the price of the stock in period t is given by:

$$(15) \quad P_t = (D_t / r) + b_t = (D_t / r) + (1+r)b_{t-1} + ce_t$$

We need to see if the right-hand side of equation (15) is equivalent to the right-hand side of equation (2), the first-order condition. Since equation (15) holds every period, we can write the price of the stock in period $t+1$ as:

$$(16) \quad P_{t+1} = (D_{t+1} / r) + (1+r)b_t + ce_{t+1} = [(D_t + e_{t+1}) / r] + (1+r)b_t + ce_{t+1}$$

where the last step uses $D_{t+1} = D_t + e_{t+1}$. Dividing both sides of equation (16) by $(1+r)$ and taking the time t expectation of the resulting expression gives us:

$$(17) \quad E_t \left[\frac{P_{t+1}}{1+r} \right] = \frac{D_t}{r(1+r)} + b_t$$

Adding $E_t [D_{t+1} / (1+r)]$ to both sides of equation (17) gives us:

$$(18) \quad E_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right] = E_t \left[\frac{D_{t+1}}{1+r} \right] + \frac{D_t}{r(1+r)} + b_t = \frac{rD_t + D_t}{r(1+r)} + b_t = \frac{(1+r)D_t}{r(1+r)} + b_t$$

and thus finally:

$$(19) \quad E_t \left[\frac{D_{t+1} + P_{t+1}}{1+r} \right] = \frac{D_t}{r} + b_t$$

Therefore, the right-hand sides of equations (15) and (2) are equivalent and thus the first-order condition is satisfied. With this formulation, the innovation to dividends, e , gets built into the bubble. Thus a positive realization of e does not just raise the expected path of dividends, it also raises the path of the bubble and the current price responds to both of these changes. It is in this sense that the price of the stock overreacts to changes in dividends.

Problem 7.9

a) Suppose the individual reduces her consumption by a small (formally infinitesimal) amount dC in period t . The utility cost of doing this is equal to the marginal utility of consumption in period t , $1/C_t$, times dC .

Thus we have:

$$(1) \quad \text{utility cost} = dC / C_t$$

This reduction in consumption allows the individual to purchase dC/P_t trees in period t . In period $t + 1$, the individual will receive the extra output from her additional holdings of trees. She gets to consume an extra $[dC/P_t]Y_{t+1}$. The individual then sells her additional holdings of trees for $[dC/P_t]P_{t+1}$ and consumes the proceeds. Thus her total extra consumption in period $t + 1$ is given by $[dC/P_t]Y_{t+1} + [dC/P_t]P_{t+1}$. The marginal utility of consumption in period $t + 1$ is $1/C_{t+1}$. Thus the expected, discounted utility benefit from this action is:

$$(2) \text{ expected utility benefit} = E_t \left[\frac{1}{1+\rho} \frac{1}{C_{t+1}} \left(\frac{dC}{P_t} Y_{t+1} + \frac{dC}{P_t} P_{t+1} \right) \right]$$

If the individual is optimizing, a marginal change of this type must leave expected utility unchanged. This means that the utility cost must equal the expected utility benefit, or:

$$(3) \frac{dC}{C_t} = E_t \left[\frac{1}{1+\rho} \frac{1}{C_{t+1}} \frac{dC}{P_t} (Y_{t+1} + P_{t+1}) \right]$$

Canceling the dC 's (which is somewhat informal) gives us:

$$(4) \frac{1}{C_t} = E_t \left[\frac{1}{1+\rho} \frac{1}{C_{t+1}} \frac{1}{P_t} (Y_{t+1} + P_{t+1}) \right]$$

We can now solve equation (4) for P_t in terms of Y_t and expectations involving Y_{t+1} , P_{t+1} and C_{t+1} . Note that we can replace C_t with Y_t and that P_t is not uncertain at time t . Using these facts, equation (4) can be rewritten as:

$$(5) \frac{1}{Y_t} = \frac{1}{P_t} E_t \left[\frac{1}{1+\rho} \frac{1}{C_{t+1}} (Y_{t+1} + P_{t+1}) \right]$$

Solving equation (5) for the price of a tree in period t gives us:

$$(6) P_t = \frac{Y_t}{1+\rho} E_t \left[\frac{Y_{t+1} + P_{t+1}}{C_{t+1}} \right]$$

b) Since $C_{t+s} = Y_{t+s}$ for all $s \geq 0$, equation (6) can be written as:

$$(7) P_t = \frac{Y_t}{1+\rho} E_t \left[1 + \frac{P_{t+1}}{Y_{t+1}} \right] = \frac{Y_t}{1+\rho} + \frac{Y_t}{1+\rho} E_t \left[\frac{P_{t+1}}{Y_{t+1}} \right]$$

Equation (7) holds for all periods and so we can write the price of a tree in period $t + 1$ as:

$$(8) P_{t+1} = \frac{Y_{t+1}}{1+\rho} + \frac{Y_{t+1}}{1+\rho} E_{t+1} \left[\frac{P_{t+2}}{Y_{t+2}} \right]$$

Substituting equation (8) into equation (7) yields:

$$(9) P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{1+\rho} E_t \left[\frac{1}{1+\rho} + \frac{1}{1+\rho} E_{t+1} \left(\frac{P_{t+2}}{Y_{t+2}} \right) \right]$$

Now use the law of iterated projections -- for any variable x , $E_t E_{t+1} x_{t+2} = E_t x_{t+2}$ -- to obtain:

$$(10) P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{(1+\rho)^2} + \frac{Y_t}{(1+\rho)^2} E_t \left[\frac{P_{t+2}}{Y_{t+2}} \right]$$

After repeated substitutions, we will have:

$$(11) P_t = \frac{Y_t}{1+\rho} + \frac{Y_t}{(1+\rho)^2} + \dots + \frac{Y_t}{(1+\rho)^s} + \frac{Y_t}{(1+\rho)^s} E_t \left[\frac{P_{t+s}}{Y_{t+s}} \right]$$

Imposing the no-bubbles condition that $\lim_{t \rightarrow \infty} E_t [(P_{t+s}/Y_{t+s})/(1+\rho)^s] = 0$, the price of a tree in period t can be written as:

$$(12) \quad P_t = Y_t \left[\frac{1}{1+\rho} + \frac{1}{(1+\rho)^2} + \dots \right]$$

Since $1/(1+\rho) < 1$, the sum converges and we can write:

$$(13) \quad P_t = Y_t \left[\frac{1/(1+\rho)}{1 - [1/(1+\rho)]} \right] = Y_t \left[\frac{1/(1+\rho)}{\rho/(1+\rho)} \right]$$

Thus, finally, the price of a tree in period t is:

$$(14) \quad P_t = Y_t / \rho$$

c) There are two effects of an increase in the expected value of dividends at some future date. The first is the fact that at a given marginal utility of consumption, the higher expected dividends increase the attractiveness of owning trees. This tends to raise the current price of a tree. However, since consumption equals dividends in this model, higher expected dividends in that future period means higher consumption and thus lower marginal utility of consumption in that future period. This tends to reduce the attractiveness of owning trees — the tree is going to pay off more in a time when marginal utility is expected to be low — and thus tends to lower the current price of a tree. In the case of logarithmic utility, these two forces exactly offset each other, leaving the current price of a tree unchanged in the face of a rise in expected future dividends.

d) The path of consumption is equivalent to the path of output. Thus if output follows a random walk, so does consumption. But if output does not follow a random walk, then consumption does not either.

Problem 7.10

a) Suppose the individual reduces her holdings of the good-state asset by a small (formally, infinitesimal) amount dA_G . This change means that if the good state occurs — which it will, with probability $1/2$ — the individual loses dA_G times the marginal utility of consumption in the good state, which is $U'(1)$. Thus:

$$(1) \quad \text{expected utility loss} = U'(1)dA_G/2$$

Since p represents the relative price of the bad-state asset to the good-state asset, selling dA_G of the good-state asset allows the individual to purchase dA_G/p of the bad-state asset. This means that if the bad state occurs — which it will, with probability $1/2$ — the individual gains dA_G/p times the expected marginal utility of consumption in the bad state. Fraction λ of the population consumes $1 - (\phi/\lambda)$ in the bad state, while fraction $(1 - \lambda)$ consumes one. Thus the expected marginal utility of consumption in the bad state is $\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)$. Putting all of this together, we have:

$$(2) \quad \text{expected utility benefit} = [dA_G/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]/2$$

If the individual is optimizing, this change in holdings of the two assets must leave expected utility unchanged. Thus the expected utility loss must equal the expected utility gain, or:

$$(3) \quad U'(1)dA_G/2 = [dA_G/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]/2$$

b) From equation (3), canceling the $1/2$'s and the dA_G 's (which is somewhat informal), we have:

$$(4) \quad U'(1) = [1/p][\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)]$$

Solving equation (4) for p gives us:

$$(5) \quad p = \frac{\lambda U'(1 - (\phi/\lambda)) + (1 - \lambda)U'(1)}{U'(1)}$$

c) The change in the equilibrium relative price of the bad-state asset to the good-state asset due to a change in λ is:

$$(6) \quad \frac{\partial p}{\partial \lambda} = U'(1 - (\phi/\lambda)) + \lambda U''(1 - (\phi/\lambda))(\phi/\lambda^2) - U'(1)$$

which simplifies to:

$$(7) \quad \frac{\partial p}{\partial \lambda} = U'(1 - (\phi/\lambda)) - U'(1) + U''(1 - (\phi/\lambda))(\phi/\lambda)$$

d) If utility is quadratic, then $U'(C)$ is a linear function of C since $U''(C)$ is a constant. See the diagram at right. We can calculate the slope of the $U'(C)$ line as:

$$(8) \quad \text{slope} = \frac{U'(1 - (\phi/\lambda)) - U'(1)}{1 - (\phi/\lambda) - 1}$$

or:

$$(9) \quad \text{slope} = \frac{U'(1 - (\phi/\lambda)) - U'(1)}{-\phi/\lambda}$$

We also know that the slope of this line is $U''(C)$ at any value of C and in particular it is equal to $U''(1 - (\phi/\lambda))$. Equating these two expressions for the slope gives us:

$$(10) \quad \frac{U'(1 - (\phi/\lambda)) - U'(1)}{-\phi/\lambda} = U''(1 - (\phi/\lambda))$$

and hence:

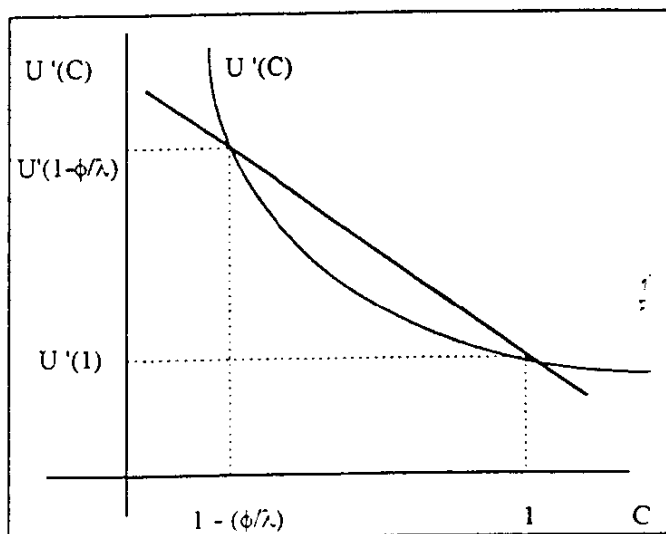
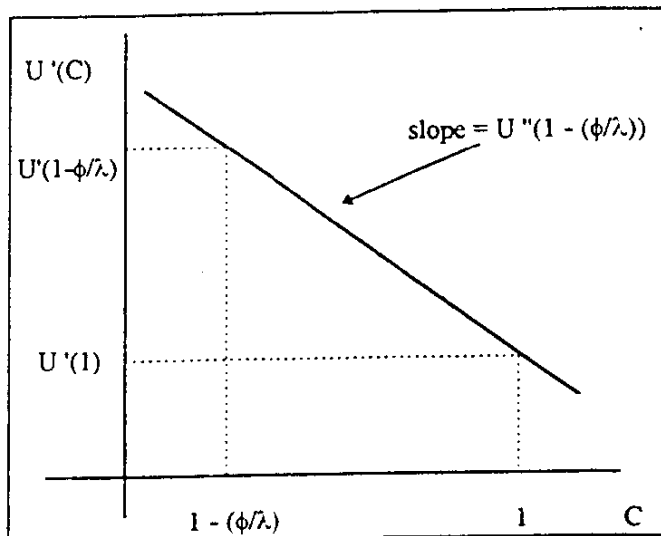
$$(11) \quad U'(1 - (\phi/\lambda)) - U'(1) + (\phi/\lambda)U''(1 - (\phi/\lambda)) = 0$$

The left-hand side of equation (11) is $\partial p / \partial \lambda$ and thus it equals zero as required. With quadratic utility, a marginal change in the concentration of aggregate shocks has no effect on the relative price of the bad-state asset to the good-state asset.

e) If $U'''(\bullet) > 0$ everywhere, then $U'(C)$ is a convex function of C — $U''(C)$ rises, or becomes less negative, as C rises. Just as in part (d), we can draw in a straight line that goes through $[1 - \phi/\lambda, U'(1 - \phi/\lambda)]$ as well as $[1, U'(1)]$. Again, it will have slope equal to:

$$(9) \quad \text{slope} = \frac{U'(1 - (\phi/\lambda)) - U'(1)}{-\phi/\lambda}$$

However, from the diagram, we can see that the slope of this line is less negative, or greater, than the slope of $U'(C)$ at $1 - (\phi/\lambda)$, which is $U''(1 - (\phi/\lambda))$. That is, we have:



$$(12) \frac{U'(1 - (\phi/\lambda)) - U'(1)}{-\phi/\lambda} > U''(1 - (\phi/\lambda))$$

or simply:

$$(13) U'(1 - (\phi/\lambda)) - U'(1) + (\phi/\lambda)U''(1 - (\phi/\lambda)) < 0$$

The left-hand side of equation (13) $\partial p / \partial \lambda$ and thus $\partial p / \partial \lambda < 0$. If $U'''(\bullet)$ is everywhere positive, a fall in λ causes p to rise. That is, a marginal increase in the concentration of the aggregate shock — a fall in λ — causes the relative price of the bad-state asset to the good-state asset to rise.

SOLUTIONS TO CHAPTER 8

Problem 8.1

a) Given K and the fixed quantity demanded, Y , the firm will hire enough labor to meet that demand. Given the production function:

$$(1) Y = K^\alpha L^{1-\alpha}$$

the firm will hire:

$$(2) L = Y^{1/(1-\alpha)} K^{-\alpha/(1-\alpha)}$$

b) Substituting equation (2) for the firm's choice of L into the profit function $-\pi = PY - WL - r_K K$ — we have:

$$(3) \pi = PY - W[Y^{1/(1-\alpha)} K^{-\alpha/(1-\alpha)}] - r_K K$$

c) The first-order condition for the firm's choice of K is:

$$(4) \frac{\partial \pi}{\partial K} = \frac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{[-\alpha/(1-\alpha)]-1} - r_K = 0$$

or simply:

$$(5) \frac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{-1/(1-\alpha)} = r_K$$

In order for the value of K in equation (5) to be a maximum, we require $\partial^2 \pi / \partial K^2$ to be negative. This derivative is:

$$(6) \frac{\partial^2 \pi}{\partial K^2} = \left(\frac{-1}{1-\alpha} \right) \frac{\alpha}{1-\alpha} W Y^{1/(1-\alpha)} K^{(\alpha-2)/(1-\alpha)} < 0$$

So the second-order condition is satisfied since $\alpha < 1$.

d) Solving equation (5) for K gives us:

$$(7) K^{1/(1-\alpha)} = \left(\frac{\alpha}{1-\alpha} \right) \frac{W Y^{1/(1-\alpha)}}{r_K}$$

Taking both sides of equation (7) to the exponent $(1-\alpha)$ gives us the firm's choice of K :

$$(8) K = Y \left(\frac{\alpha}{1-\alpha} \right)^{1-\alpha} \left(\frac{W}{r_K} \right)^{(1-\alpha)}$$

Thus changes in the price of the firm's product do not directly affect the profit-maximizing choice of K , although changes in P likely change Y . The elasticity of K with respect to the wage, W , is $(1-\alpha)$, which is positive. Its elasticity with respect to the rental price of capital, r_K , is $-(1-\alpha)$, which is negative. Finally, the elasticity of K with respect to the quantity demanded is one.

Problem 8.2

a) At each point in time from t to $t+T$, the firm is allowed to deduct P_K/T from its taxable income. This allows it to save $\tau(P_K/T)$ in taxes at each point in time from t to $t+T$, where τ is the marginal tax rate. If i is the constant interest rate, the present value of the reduction in the firm's corporate tax liabilities is given by the following expression.

$$(1) \text{tax saving} = \int_{s=t}^{t+T} e^{-i(s-t)} \tau(P_K/T) ds$$

which implies:

$$(2) \text{ tax saving} = \tau(P_K/T) \int_{s=t}^{t+T} e^{-i(s-t)} ds = \tau(P_K/T) \left[\frac{-1}{i} e^{-i(s-t)} \right]_{s=t}^{s=t+T} = \tau(P_K/T) \left[\frac{1 - e^{-iT}}{i} \right]$$

Since the after-tax price of the capital good is its pretax price minus the present value of the tax saving that results, we have:

$$(2) P_K^{AT} = P_K - \tau(P_K/T) \left[\frac{1 - e^{-iT}}{i} \right] = P_K \left[1 - (\tau/T) \left(\frac{1 - e^{-iT}}{i} \right) \right]$$

where P_K^{AT} denotes the after-tax price of the capital good. To get a feel for the magnitudes involved, if $\tau = 0.20$, $P_K = 1000$, $T = 10$ and $i = 0.05$, P_K^{AT} is about 843. This means that the tax saving is approximately 16.7 percent of the purchase price of the capital good.

b) Substituting $i = r + \pi$ into equation (2) gives us:

$$(3) P_K^{AT} = P_K \left[1 - (\tau/T) \left(\frac{1 - e^{-(r+\pi)T}}{r + \pi} \right) \right]$$

A rise in π , with r fixed, increases i and thus reduces the present value of the tax savings from purchasing the capital good. This means that it raises the after-tax price of the capital good to the firm. For example, with the numbers from part (a), if π rises so that $i = 0.10$ rather than $i = 0.05$, the after tax price of the capital good is about 874. Thus the tax saving is now only about 13.6 percent of the purchase price of the capital good.

Problem 8.3

From equation (8.4) in the text, the real user cost of capital is:

$$(1) r_K(t) = [r(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t)$$

where $r(t)$ is the relevant real interest rate, δ is the rate of depreciation and $p_K(t)$ is the real price of capital. Here, the capital we are talking about is owner-occupied housing. The after-tax real interest rate for owner-occupied housing is $r(t) - \tau i(t)$ where τ is the marginal tax rate. This is due to the fact that nominal interest payments are tax deductible.

Intuitively, if an individual foregoes selling her home, she does lose $r(t)p_K(t)$ — the interest she could obtain by selling it and saving the proceeds — but she does get the bonus of deducting her nominal interest payments from her income. Thus she receives τ times $i(t)p_K(t)$ in tax savings by holding on to her home, which reduces the user cost of capital. Thus for owner-occupied housing, equation (1) becomes:

$$(2) r_K(t) = [r(t) - \tau i(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t)$$

Substituting $i(t) = r(t) + \pi(t)$ into equation (2) gives us:

$$(3) r_K(t) = [r(t) - \tau r(t) - \tau \pi(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t)$$

which implies:

$$(4) r_K(t) = [(1 - \tau)r(t) - \tau \pi(t) + \delta - (\dot{p}_K(t)/p_K(t))]p_K(t)$$

To see how an increase in inflation for a given real interest rate affects $r_K(t)$, take the derivative of $r_K(t)$ with respect to $\pi(t)$:

$$(5) \partial r_K(t) / \partial \pi(t) = -\tau p_K(t) < 0$$

An increase in inflation reduces the user cost of owner-occupied housing since it allows the homeowner more in the way of tax deductions on the nominal interest payments. Thus an increase in inflation increases the desired stock of owner-occupied housing.

Problem 8.4

a) The planner's problem is to maximize the discounted value of lifetime utility for the representative household, which is given by:

$$(1) \quad U = \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \quad \beta \equiv \rho - n - (1-\theta)g$$

subject to the capital-accumulation equation:

$$(2) \quad \dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t)$$

The control variable is the variable that can be controlled freely by the planner, which is consumption per unit of effective labor, $c(t)$. The state variable is the variable whose value at any time is determined by past decisions of the planner, which is capital per unit of effective labor, $k(t)$. Finally, the shadow value of the state variable is the costate variable, which we will denote $\mu(t)$.

The current value Hamiltonian is thus:

$$(3) \quad H(k(t), c(t)) = \frac{c(t)^{1-\theta}}{1-\theta} + \mu(t)[f(k(t)) - c(t) - (n+g)k(t)]$$

b) The first condition characterizing the optimum is that the derivative of the Hamiltonian with respect to the control variable at each point is zero, or:

$$(4) \quad \frac{\partial H(k(t), c(t))}{\partial c(t)} = c(t)^{-\theta} - \mu(t) = 0$$

The second condition is that the derivative of the Hamiltonian with respect to the state variable equals the discount rate times the costate variable minus the derivative of the costate variable with respect to time, or:

$$(5) \quad \frac{\partial H(k(t), c(t))}{\partial k(t)} = \mu(t)f'(k(t)) - \mu(t)(n+g) = \beta\mu(t) - \dot{\mu}(t)$$

The final condition is the transversality condition. The limit as t goes to infinity of the discounted value of the costate variable times the state variable equals zero, or:

$$(6) \quad \lim_{t \rightarrow \infty} e^{-\beta t} \mu(t)k(t) = 0$$

c) From equation (4) we have:

$$(7) \quad \mu(t) = c(t)^{-\theta}$$

Taking the time derivative of both sides of equation (7) yields:

$$(8) \quad \dot{\mu}(t) = -\theta c(t)^{-\theta-1} \dot{c}(t) = -\theta c(t)^{-\theta} \frac{\dot{c}(t)}{c(t)}$$

From equation (5), we have:

$$(9) \quad \dot{\mu}(t) = \mu(t)[\beta - f'(k(t)) + (n+g)]$$

Equating these two expressions for $\dot{\mu}(t)$ gives us:

$$(10) \quad -\theta c(t)^{-\theta} \frac{\dot{c}(t)}{c(t)} = \mu(t)[\beta - f'(k(t)) + (n+g)]$$

Substituting equation (7) for $\mu(t)$ and $f'(k(t)) = r(t)$ into equation (10) yields:

$$(11) \quad -\theta c(t)^{-\theta} \frac{\dot{c}(t)}{c(t)} = c(t)^{-\theta} [\beta - r(t) + (n+g)]$$

Canceling the $c(t)^{-\theta}$, dividing both sides by $-\theta$ and substituting for $\beta \equiv \rho - n - (1-\theta)g$ gives us:

$$(12) \quad \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - (n+g) - \rho + n + (1-\theta)g}{\theta}$$

which simplifies to:

$$(13) \quad \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

Equation (13) is identical to the Euler equation in the decentralized equilibrium. See equation (2.19) in the text.

d) Dividing both sides of equation (9) by $\mu(t)$ leaves us with:

$$(14) \quad \frac{\dot{\mu}(t)}{\mu(t)} = \beta + (n + g) - r(t)$$

where we have used $f'(k(t)) = r(t)$. Note that equation (14) can be written as:

$$(15) \quad \partial \ln \mu(t) / \partial t = \beta + (n + g) - r(t)$$

Integrating both sides of equation (15) from time $\tau = 0$ to time $\tau = t$ gives us:

$$(15) \quad \ln \mu(t) - \ln \mu(0) = [\beta + (n + g)] \tau \Big|_{\tau=0}^{\tau=t} - \int_{\tau=0}^t r(\tau) d\tau$$

Using the definition of $R(t)$ and simplifying gives us:

$$(16) \quad \ln \mu(t) = \ln \mu(0) + \beta t + (n + g)t - R(t)$$

Taking the exponential function of both sides of equation (16) yields:

$$(17) \quad \mu(t) = \mu(0) e^{\beta t} e^{(n+g)t} e^{-R(t)}$$

Thus $e^{-\beta t} \mu(t)$ is proportional to $e^{-R(t)} e^{(n+g)t}$.

This implies that the transversality condition — equation (6) — is equivalent to:

$$(18) \quad \lim_{t \rightarrow \infty} e^{-R(t)} e^{(n+g)t} k(t) = 0$$

From equation (2.12) in the text, the household's budget constraint, expressed in terms of limiting behavior, is given by:

$$(19) \quad \lim_{t \rightarrow \infty} e^{-R(t)} e^{(n+g)t} k(t) \geq 0$$

Comparing equations (18) and (19), we can see that the transversality condition will hold if and only if the budget constraint is met with equality. Thus we have shown that the solution to the social planner's problem in the Ramsey model is the same as the decentralized equilibrium. Hence that decentralized equilibrium must be Pareto efficient.

Problem 8.5

The equation of motion for the market value of capital, q , is:

$$(1) \quad \dot{q}(t) = r q(t) - \pi(K(t))$$

where $\pi'(\bullet) < 0$. The condition required for $\dot{q} = 0$ is given by:

$$(2) \quad q = \pi(K)/r$$

The equation of motion for capital, K , is:

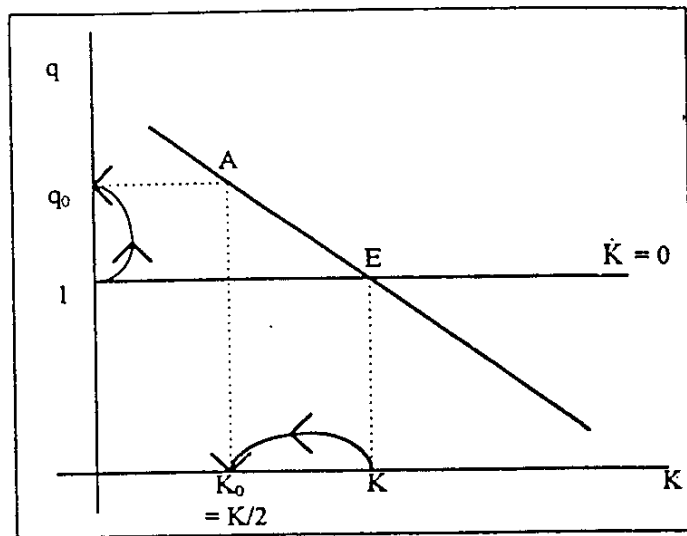
$$(3) \quad \dot{K}(t) = f(q(t))$$

where $f(q) = NC^{-1}(q - 1)$ with $f(1) = 0$ and $f'(\bullet) > 0$. The condition required for $\dot{K} = 0$ is given by:

$$(4) \quad q = 1$$

a) The destruction of half of the capital stock does not cause either the $\dot{K} = 0$ or the $\dot{q} = 0$ loci to shift. Both of these are already drawn allowing for K to vary. At the time of the destruction, K falls to $K_0 = K/2$.

For the economy to return to a stable equilibrium, q must adjust so that the economy is on the saddle path. Thus q must jump up to q_0 , putting the economy at point A in the diagram at right. Intuitively, since profits are higher at the lower K , the capital that is left is more valuable and so the market value of capital is now higher.

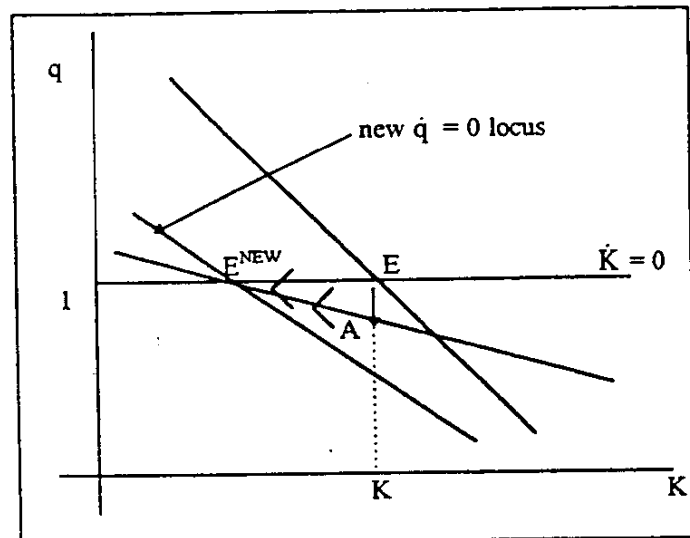


The economy then moves down the saddle path with q falling and K rising. Intuitively, the higher market value of capital attracts investment and so the capital stock begins to build back up. As it does so, profits begin to fall and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is back at its original level. Hence the economy eventually returns to point E.

b) Profits at a given K are now $(1 - \tau)\pi(K)$ rather than $\pi(K)$. The condition required for $\dot{q} = 0$ is now given by:

$$(5) \quad q = (1 - \tau)\pi(K)/r$$

At a given K , the value of q that makes $\dot{q} = 0$ is now lower so the new $\dot{q} = 0$ locus lies below the old one. In addition, the slope of the $\dot{q} = 0$ locus is $\partial q / \partial K = (1 - \tau)\pi'(K)/r$ rather than $\pi'(K)/r$. With $(1 - \tau) < 1$, this new slope is less negative and so the $\dot{q} = 0$ locus becomes flatter. The $\dot{K} = 0$ locus is unaffected. See the diagram at right.



K , the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump down so that the economy is on the new saddle path at point A. Intuitively, since the government is now taking a fraction of profits, existing capital is less valuable and so the market value of capital falls. The economy then moves up the new saddle path with K falling and q rising. Intuitively, the lower market value of capital discourages investment and so the capital stock begins falling. As it does so, profits begin to rise back up and thus so does the market value of capital. This process continues until the market value of capital returns to its long-run-equilibrium value of one and the capital stock is at a permanently lower level. The economy winds up at point E^{NEW} in the diagram. The lower capital and thus higher pretax profits offset the fact that the government takes a fraction of those profits.

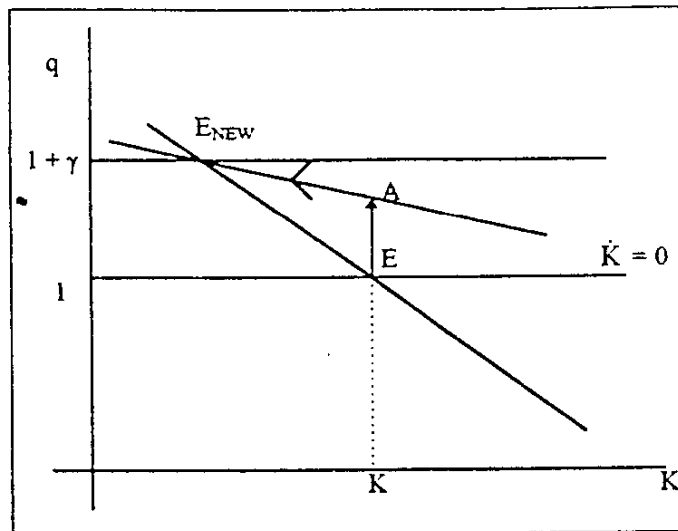
c) One of the conditions required for optimization is that the firm invests to the point where the cost of acquiring capital equals the value of that capital, q . With this tax on investment, the cost of acquiring a unit of capital is the purchase price (which is fixed at one) plus the tax, γ , plus the marginal adjustment cost, $C'(I)$. Thus analogous to equation (8.18) in the text, we now have:

$$(6) \quad 1 + \gamma + C'(I(t)) = q(t)$$

Since $C'(0)$ is zero, equation (6) implies that $I(t)$ is zero (and thus $\dot{K} = 0$) when $q(t) = 1 + \gamma$. So the equation of the $\dot{K} = 0$ locus is now:

$$(7) \quad q = 1 + \gamma$$

Thus an investment tax of γ shifts the $\dot{K} = 0$ locus up by γ . The $\dot{q} = 0$ locus is unaffected. See the diagram.



K , the stock of capital, cannot jump at the time of the implementation of the tax. Thus q must jump up so that the economy is on the new saddle path at point A. Intuitively, because the tax will reduce investment, it means that the industry's profits (neglecting the tax) will eventually be higher, and thus that existing capital is more valuable. The economy then moves up the new saddle path until it reaches point E_{NEW} . The capital stock is permanently lower and the pretax market value of capital is equal to $1 + \gamma$ — the after-tax market value is again equal to one.

Problem 8.6

The important point is that q is anticipated to jump up discontinuously at the time of the capital levy, time T . Consider what is required, if there is a market for shares in firms, for individuals to be willing to hold those shares through the interval where the one-time tax on capital holdings is imposed. Consider the market value of capital an instant, ϵ , before the levy and an instant after the levy and then look at what happens as ϵ goes to zero. The key point is that the market value of capital an instant before the levy, $q(T - \epsilon)$, must equal $(1 - f)$ times the market value of capital an instant after the levy. If it did not, — in light of the levy — holders of shares in firms would be expecting capital losses that they could avoid.

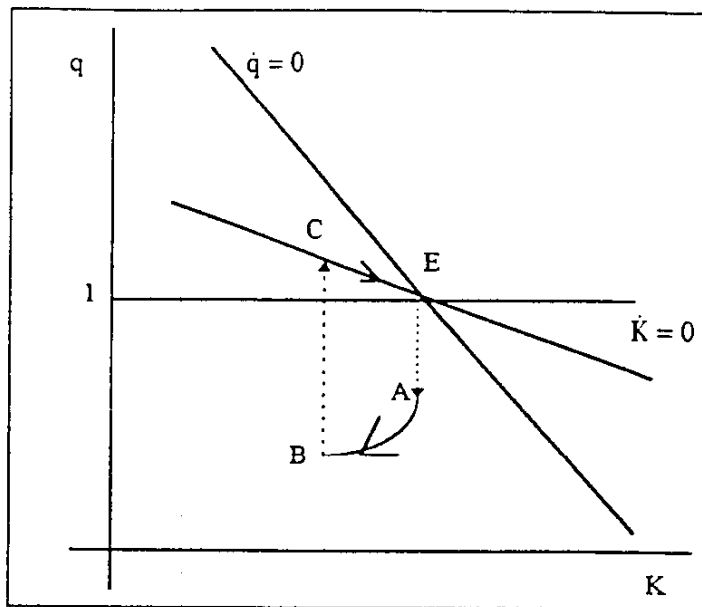
Therefore, $q(T - \epsilon)$ must equal $(1 - f)q(T + \epsilon)$ or:

$$(1) \quad q(T - \epsilon)/q(T + \epsilon) = (1 - f)$$

For example, if $f = 0.10$ or ten percent, then the value of q an instant before the levy must equal 90 percent of its value an instant after the levy. Thus at time T , q jumps up to close that 10 percent gap. In addition, that jump must put the economy somewhere on the saddle path in order for the economy to return to a stable equilibrium.

Thus at the time of the news, q must jump down, putting the economy at a point like A in the diagram at right. The economy is then in a region where both q and K are falling. Thus between the time of the news and the time the levy is imposed, the market value of capital and the capital stock are falling. Intuitively, firms begin decumulating capital in anticipation of the one-time levy.

Point A must be chosen so that at the time of the levy, q can jump up by the required amount discussed above and that required jump must put the economy right on the saddle path. The stock of capital does not jump at the time of the levy. Thus at time



T, the economy jumps from a point like B to a point like C where $q_B/q_C = (1 - f)$.

After the time of the levy, the economy moves down the saddle path, eventually returning to the original equilibrium at point E. Intuitively, once the one-time tax is over with, since K is lower, profits are higher and so investment is attractive once again. Thus the capital stock begins rising back to its initial level.

Problem 8.7

a) The evolution of the stock of housing is given by:

$$(1) \dot{H} = I(p_H) - \delta H$$

Thus the condition required for $\dot{H} = 0$ is given by $I(p_H) = \delta H$. That is, in order for the stock of housing to remain constant, new investment in housing (which is an increasing function of the real price of housing) must exactly offset depreciation of the existing housing stock. Differentiating both sides of this expression with respect to H gives us the slope of the $\dot{H} = 0$ locus:

$$(2) I'(p_H) dp_H / dH = \delta$$

or:

$$(3) dp_H / dH = \delta / I'(p_H) > 0$$

Since $I'(p_H) > 0$, the $\dot{H} = 0$ locus is upward sloping in (H, p_H) space.

Rental income plus capital gains must equal the exogenous rate of return, r , or:

$$(4) \frac{R(H) + \dot{p}_H}{p_H} = r$$

Solving equation (4) for $\dot{p}_H$ yields:

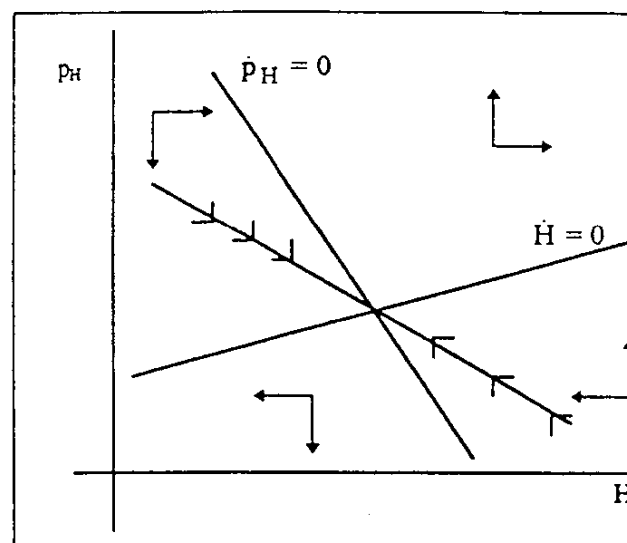
$$(5) \dot{p}_H = r p_H - R(H)$$

Therefore the condition required for $\dot{p}_H = 0$ is $r p_H - R(H) = 0$ or $p_H = R(H)/r$. Differentiating both sides of this expression with respect to H gives us the slope of the $\dot{p}_H = 0$ locus:

$$(6) dp_H / dH = R'(H)/r$$

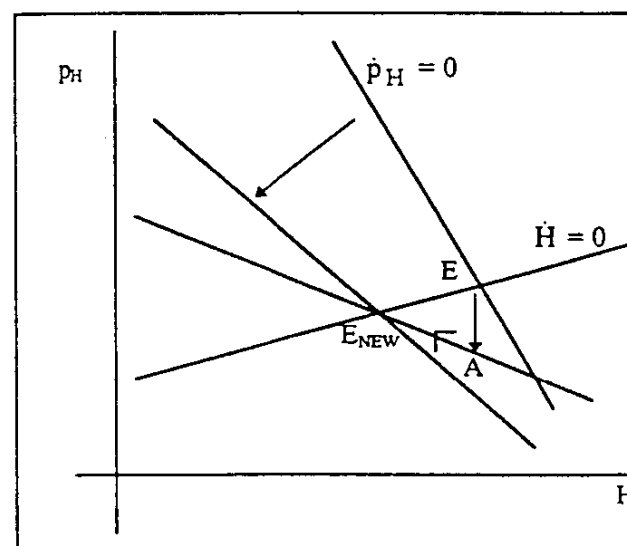
Since $R'(H) < 0$ — rent is a decreasing function of the stock of housing — the $\dot{p}_H = 0$ locus is downward sloping in (H, p_H) space.

b) Since $I'(p_H) > 0$, then from equation (1), $\dot{H}$ is increasing in p_H . This means that above the $\dot{H} = 0$ locus, $\dot{H} > 0$ and so H is rising. Intuitively, at a given H , if p_H is higher than the price necessary to keep the stock of housing constant, investment (which is an increasing function of p_H) is higher than necessary to offset depreciation. Thus the stock of housing is rising above the $\dot{H} = 0$ locus. Similarly, below the $\dot{H} = 0$ locus, $\dot{H} < 0$ and so H is falling. Intuitively, p_H and thus investment are too low to offset depreciation and keep the stock of housing constant at a given H . Thus the stock of housing is falling below the $\dot{H} = 0$ locus.



Since $R'(H) < 0$, then from equation (5), $\dot{p}_H$ is increasing in H . This means that to the right of the $\dot{p}_H = 0$ locus, $\dot{p}_H > 0$ and so p_H is rising. Intuitively, at a given p_H , if H is higher – and thus rent lower – than the level necessary to keep the price of housing constant, this lower rent must be offset by capital gains – a rising p_H – if investors are to earn the required exogenous return of r . Similarly, to the left of the $\dot{p}_H = 0$ locus, $\dot{p}_H < 0$ and so p_H is falling. If H is low – and thus rent higher – than the level necessary to keep the price of housing constant, this higher rent must be offset by capital losses in order for investors to earn the rate of return r .

c) The $\dot{p}_H = 0$ locus is defined by $p_H = R(H)/r$. A rise in r means that the p_H that makes $\dot{p}_H = 0$ is now lower at a given H . Thus the new $\dot{p}_H = 0$ locus lies below the old one. In addition, the slope of the $\dot{p}_H = 0$ locus is $R'(H)/r$ and so the rise in r makes the slope less negative. Thus the new $\dot{p}_H = 0$ locus is flatter than the old one. The $\dot{H} = 0$ locus is defined by $I(p_H) = \delta H$. Since r does not appear in this equation, the $\dot{H} = 0$ locus is unaffected.



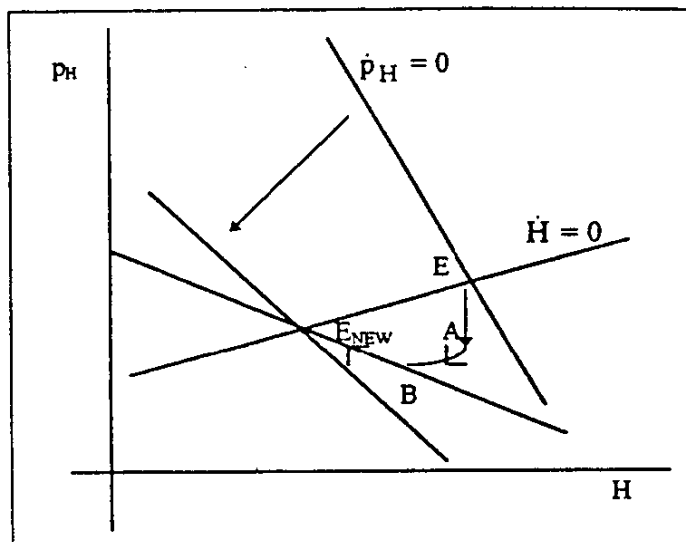
At the time of the increase in r , H – the stock of existing housing – cannot jump discontinuously. The real price of housing, p_H , must jump down to put the economy on the new saddle path. In the diagram, at the time of the rise in r , the economy jumps from point E to point A .

The discontinuous downward jump in p_H causes the amount of investment to jump down. Thus investment is no longer enough to offset depreciation at the initial value of H – we are below the $\dot{H} = 0$ locus – and the stock of housing begins to fall. As H begins to fall, rent begins to rise since $R'(H) < 0$. As the economy moves up the new saddle path, the real price of housing is rising. This means that investment is rising back up since $I'(p_H) > 0$. The economy eventually reaches point E_{NEW} where p_H is constant at a new level.

lower level and thus investment is constant at a new lower level. In addition, the stock of housing is constant at a new lower level. Finally, rent is higher.

d) An important point is that the dynamics of the system are still governed by the original $\dot{p}_H = 0$ and $\dot{H} = 0$ loci until the actual time of the increase in r . At the time of the change, H cannot jump and, importantly, neither can p_H . If p_H did jump, an instant before the rise in r , people would be expecting capital gains or losses that could be arbitrated away. Thus right at the time of the increase in r , the economy must be somewhere on the new saddle path.

At the time of the news, since the stock of housing, H , cannot jump, p_H must jump down so that the economy is at a point like A in the diagram at right. The economy is then below the $\dot{H} = 0$ locus and so H is falling. Intuitively, the price of housing is now lower and thus so is investment. It is no longer high enough to offset depreciation at the initial level of H and so H begins to fall. The economy is also to the left of the $\dot{p}_H = 0$ locus – recall, this locus does not shift until the rise in r actually occurs – and so p_H is falling. Thus between the time of the news and the time of the increase in r , H is falling and so rent, R , is increasing. In addition, p_H is falling and thus investment, which jumped down initially, continues to fall.



Right at the time of the increase in r , the $\dot{p}_H = 0$ locus shifts to the left and becomes flatter. The economy is at a point like B on the new saddle path. The economy is still below the $\dot{H} = 0$ locus and thus the stock of housing continues to fall while rent, R , continues to rise. However, the economy is now to the right of the relevant $\dot{p}_H = 0$ locus and so the real price of housing begins to rise and thus so does investment. The economy moves up the new saddle path until it reaches a new long-run equilibrium at point E_{NEW} . At this new long-run equilibrium, H is lower, R is higher, p_H is lower and I is lower.

e) Adjustment costs are not internal in this model. There are no direct costs of building housing in this setup. Internal costs are, as the text explains, the actual cost of building the new capital (here, housing) which can include the cost of training workers to do it and so on. This model exhibits external adjustment costs. As firms undertake more investment in housing, the real price of housing adjusts so that individuals do not wish to invest or disinvest at infinite rates.

f) The $\dot{H} = 0$ locus is not horizontal because investment depends on the real price of housing, p_H . Depreciation is proportional to the stock of housing and thus is higher at higher levels of H . Therefore, to keep H constant at higher levels of H requires more investment. But in order to have more investment, the price of housing must be higher since $I'(p_H) > 0$. This means that the $\dot{H} = 0$ locus is upward sloping.

Problem 8.8

a) The costs of adjustment are now given by $C(\dot{\kappa}/\kappa)\kappa$. Since the capital-accumulation equation is given by $\dot{\kappa} = 1 - \delta\kappa$, this can be written as $C((1/\kappa) - \delta)\kappa$. The firm's profits at time t are $\pi(K(t))\kappa(t) - I(t) - C((1/\kappa(t)) - \delta)\kappa(t)$. Thus the present-value Hamiltonian is:

$$(1) \quad H(\kappa(t), I(t)) = \pi(K(t))\kappa(t) - I(t) - C\left(\frac{1}{\kappa(t)} - \delta\right)\kappa(t) + q(t)[I(t) - \delta\kappa(t)]$$

b) The first condition characterizing the optimum is that the derivative of the Hamiltonian with respect to the control variable at each point is zero. Here, the control variable is investment and thus:

$$(2) \quad \frac{\partial H(\kappa(t), I(t))}{\partial I(t)} = -1 - C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\frac{1}{\kappa(t)}\kappa(t) + q(t) = 0$$

The second condition is that the derivative of the Hamiltonian with respect to the state variable equals the discount rate times the costate variable minus the derivative of the costate variable with respect to time. Since the state variable is the capital stock, we have:

$$(3) \quad \frac{\partial H(\kappa(t), I(t))}{\partial \kappa(t)} = \pi(K(t)) - C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\left(\frac{-I(t)}{\kappa(t)^2}\right)\kappa(t) - C\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right) - \delta q(t) = r q(t) - \dot{q}(t)$$

The final condition is the transversality condition. The limit as t goes to infinity of the present value of the costate variable times the state variable must be zero. Thus, we have:

$$(4) \quad \lim_{t \rightarrow \infty} e^{-rt} q(t) \kappa(t) = 0$$

c) Equation (2) states that each firm invests to the point where the purchase price of capital plus the marginal adjustment cost equals the value of capital: $1 + C'(\dot{\kappa}/\kappa) = q$. Since $C'(\dot{\kappa}/\kappa)$ is increasing in $\dot{\kappa}/\kappa$, this condition implies that $\dot{\kappa}/\kappa$ is increasing in q . And since $C'(0)$ is zero, it also implies that $\dot{\kappa}/\kappa$ is zero when q equals one. Finally, note that since q is the same for all firms, all firms choose the same value of $\dot{\kappa}/\kappa$. Thus the growth rate of the aggregate capital stock, $\dot{K}/K$, is given by the value of $\dot{\kappa}/\kappa$ that satisfies

(2). Putting this information together, we can write:

$$(5) \quad \dot{K}(t)/K(t) = f(q(t)) \quad f(1) = 0, f'(\bullet) > 0$$

where $f(q)$ is the value of $\dot{K}/K$ that satisfies $C'(\dot{K}/K) = q - 1$: $f(q) = C'^{-1}(q - 1)$. Equation (5) implies that K is increasing when $q > 1$, decreasing when $q < 1$ and constant when $q = 1$. Thus the $\dot{K} = 0$ locus is a horizontal line at $q = 1$ when drawn in (K, q) space.

d) Rearranging equation (3) to solve for $\dot{q}(t)$ yields:

$$(6) \quad \dot{q}(t) = (r + \delta)q(t) - \left[\pi(K(t)) + C'\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\left(\frac{I(t)}{\kappa(t)}\right) - C\left(\frac{\dot{\kappa}(t)}{\kappa(t)}\right) \right]$$

To simplify this expression, note first that I/κ equals $(\dot{\kappa} + \delta\kappa)/\kappa$ or $(\dot{\kappa}/\kappa) + \delta$. In addition, as we just showed, the growth rate of the representative firm's capital stock, $\dot{\kappa}/\kappa$, is the same as the growth rate of the industry-wide capital stock, $\dot{K}/K$. Thus we can rewrite equation (6) as:

$$(7) \quad \dot{q}(t) = (r + \delta)q(t) - \left[\pi(K(t)) + C'\left(\frac{\dot{K}(t)}{K(t)}\right)\left(\frac{\dot{K}(t)}{K(t)} + \delta\right) - C\left(\frac{\dot{K}(t)}{K(t)}\right) \right]$$

We can now use $\dot{K}/K = f(q)$ — equation (5) — to substitute for $\dot{K}/K$, and then use the fact that the definition of $f(\bullet)$ implies $C'(f(q)) = q - 1$. This yields:

$$(8) \quad \dot{q}(t) = (r + \delta)q(t) - \left[\pi(K(t)) + [q(t) - 1][f(q(t)) + \delta] - C(f(q(t))) \right] \equiv G(K(t), q(t))$$

e) The condition $G(K, q) = 0$ implicitly defines the locus of points in (K, q) space for which $\dot{q}$ is zero. To see how q varies with K along this locus, we therefore implicitly differentiate this condition with respect to K . This yields:

$$(9) \quad G_K(K, q) + G_q(K, q) \frac{dq}{dK} \Big|_{\dot{q}=0} = 0$$

or

$$(10) \quad \frac{dq}{dK} \Big|_{\dot{q}=0} = \frac{-G_K(K, q)}{G_q(K, q)}$$

where subscripts denote partial derivatives and $\frac{dq}{dK} \Big|_{\dot{q}=0}$ denotes the derivative of q with respect to K along the $\dot{q} = 0$ locus.

Using equation (8) to compute the derivatives in (10) yields:

$$(11) \quad G_K(K, q) = -\pi'(K)$$

and

$$(12) \quad G_q(K, q) = (r + \delta) - [(q - 1)f'(q) + (f(q) + \delta) - C'(f(q))f'(q)] = r - f(q)$$

where we again use the fact that $C'(f(q)) = q - 1$

Substituting equations (11) and (12) into equation (10) gives us:

$$(13) \quad \frac{dq}{dK} \Big|_{\dot{q}=0} = \frac{\pi'(K)}{r - f(q)}$$

Note that $f(q)$ is zero when q equals 1. Thus the slope of the $\dot{q} = 0$ locus at the point where $q = 1$ is simply $\pi'(K)/r$. Note that at this particular point, this slope is exactly the same as the slope of the $\dot{q} = 0$ locus in the version of the model in the text, where adjustment costs took the form $C(\kappa)$.

Problem 8.9

a) One of the conditions for optimization is that the marginal revenue product of capital, $\pi(K(t))$, equals its user cost, $rq(t) - \dot{q}(t)$. Rearranging this condition gives us the equation of motion for q :

$$(1) \quad \dot{q}(t) = rq(t) - \pi(K(t))$$

Substituting the profit function -- $\pi(K) = a - bK$ -- into equation (1) gives us:

$$(2) \quad \dot{q}(t) = rq(t) - a + bK(t)$$

The $\dot{q} = 0$ locus is therefore given by:

$$(3) \quad rq - a + bK = 0$$

or solving for q as a function of K , we have:

$$(4) \quad q = (a - bK)/r$$

So the $\dot{q} = 0$ locus has a constant slope of $-b/r$.

To find the long-run-equilibrium value of K (denoted K^*), we need to find the intersection of the $\dot{q} = 0$ locus -- as given by equation (4) -- and the $\dot{K} = 0$ locus. The $\dot{K} = 0$ locus is given by $q = 1$, which means that we already know that the long-run-equilibrium value of q , q^* , is one. Substituting $q = 1$ into equation (4) and solving for K^* gives us:

$$(5) \quad K^* = (a - r)/b$$

We can now use the method of Section 2.6 to find the slope of the saddle path. We need to first solve for the equation of motion of $K(t)$. One of the conditions for optimization is that each firm invests to the point

where the purchase price of capital (which is fixed at one) plus the marginal adjustment cost equals the value of capital, q . We are assuming quadratic costs of adjustment — $C(\dot{\kappa}) = \alpha \dot{\kappa}^2 / 2$ — and thus the marginal adjustment cost is:

$$(6) \quad \partial C(\dot{\kappa}) / \partial \dot{\kappa} = \alpha \dot{\kappa}$$

Thus we have $1 + \alpha \dot{\kappa} = q$, which implies

$$(7) \quad \dot{\kappa} = (q - 1) / \alpha$$

Since q is the same for all firms, all firms choose the same value of investment, $\dot{\kappa}$. Thus the rate of change of the aggregate capital stock, K , is given by:

$$(8) \quad \dot{K} = N(q - 1) / \alpha$$

where N is the number of firms.

Since q^* and K^* are constants, we can rewrite the equations of motion — (2) and (8) — as:

$$(9) \quad (\dot{q} - \dot{q}^*) = r q - a + b K \quad \text{and} \quad (10) \quad (\dot{K} - \dot{K}^*) = N(q - 1) / \alpha$$

where $(\dot{q} - \dot{q}^*)$ is the derivative of $(q - q^*)$ with respect to time and $(\dot{K} - \dot{K}^*)$ is the derivative of $(K - K^*)$ with respect to time. Dividing both sides of equation (9) by $q - q^*$ gives us:

$$(10) \quad \frac{(\dot{q} - \dot{q}^*)}{q - q^*} = \frac{r q - a + b K}{q - q^*}$$

From equation (5), we can write:

$$(11) \quad K - K^* = (b K - a + r) / b$$

or rearranging to solve for $b K$:

$$(12) \quad b K = b(K - K^*) + a - r$$

Substituting equation (12) into equation (10) gives us:

$$(13) \quad \frac{(\dot{q} - \dot{q}^*)}{q - q^*} = \frac{r q - a + b(K - K^*) + a - r}{q - q^*} = \frac{r(q - 1)}{q - q^*} + \frac{b(K - K^*)}{q - q^*} = r + b \frac{K - K^*}{q - q^*}$$

where the last step uses the fact that $q^* = 1$.

Dividing both sides of equation (10) by $K - K^*$ and noting that $q^* = 1$, we have:

$$(14) \quad \frac{(\dot{K} - \dot{K}^*)}{K - K^*} = \frac{N}{\alpha} \left(\frac{q - q^*}{K - K^*} \right)$$

Equations (13) and (14) imply that the growth rates of $q - q^*$ and $K - K^*$ depend only on the ratio of $q - q^*$ to $K - K^*$. Given this, consider what happens if the values of q and K are such that $q - q^*$ and $K - K^*$ are falling at the same rate. This implies that the ratio of $q - q^*$ to $K - K^*$ is not changing, and thus that their growth rates are not changing. Thus $q - q^*$ and $K - K^*$ continue to fall at equal rates. In terms of a phase diagram, from a point where $q - q^*$ and $K - K^*$ are falling at equal rates, the economy simply moves along a straight line saddle path to (K^*, q^*) with the distance from (K^*, q^*) falling at a constant rate.

Let μ denote $(\dot{K} - \dot{K}^*) / (K - K^*)$. Then equation (14) implies:

$$(15) \quad \mu = \frac{N}{\alpha} \left(\frac{q - q^*}{K - K^*} \right)$$

or solving for the ratio of $q - q^*$ to $K - K^*$:

$$(16) \quad \frac{q - q^*}{K - K^*} = \frac{\alpha \mu}{N}$$

From equation (13), the condition that $(\dot{q} - \dot{q}^*) / (q - q^*)$ always equals $(\dot{K} - \dot{K}^*) / (K - K^*)$ is thus:

$$(17) \quad \mu = r + (bN / \alpha \mu)$$

or:

$$(18) \quad \alpha \mu^2 - \alpha r \mu - bN = 0$$

Using the quadratic formula to solve for μ yields:

$$(19) \mu = \frac{\alpha r \pm \sqrt{\alpha^2 r^2 + 4\alpha bN}}{2\alpha} = \frac{r \pm \sqrt{r^2 + (4bN/\alpha)}}{2}$$

If μ is positive, then $q(t) - q^*$ and $K(t) - K^*$ are growing. That is, instead of moving along a straight line toward (K^*, q^*) , the economy is moving on a straight line away from (K^*, q^*) . Thus μ must be negative and hence:

$$(20) \mu_1 = \frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2}$$

Thus equation (16) with $\mu = \mu_1$ tells us how q and K must be related on the saddle path. Substituting equation (20) into equation (16) gives us:

$$(21) \frac{q - q^*}{K - K^*} = \frac{\alpha \left[r - \sqrt{r^2 + (4bN/\alpha)} \right]}{2N}$$

or solving for q as a function of K :

$$(22) q = q^* + \alpha \left[\frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2N} \right] (K - K^*)$$

Thus, the slope of the saddle path is:

$$(23) \left. \frac{\partial q}{\partial K} \right|_{sp} = \alpha \left[\frac{r - \sqrt{r^2 + (4bN/\alpha)}}{2N} \right] < 0$$

Problem 8.10

a) Consider the situation where $a(t + \tau)$ is certain to equal $E_t[a(t + \tau)]$ for all $\tau \geq 0$ so that there is no uncertainty. The value of q at some date $t + \tau$ can then be written as the value of q in period t plus the "sum" of all the changes in q from time t to time $t + \tau$. More formally:

$$(1) \hat{q}(t + \tau, t) = q(t) + \int_{s=t}^{\tau} \dot{\hat{q}}(t + s, t) ds$$

where $\hat{q}(t + \tau, t)$ is notation meaning the path of q when a is certain to equal its expected value. Since $\dot{q} = r q - \pi(K)$, then with this particular profit function we have $\dot{q} = r q - a + bK$. Thus equation (1) can be written as:

$$(2) \hat{q}(t + \tau, t) = q(t) + \int_{s=t}^{\tau} [r \hat{q}(t + s, t) - E_t[a(t + s)] + b \hat{K}(t + s, t)] ds$$

where we have substituted in for $a(t + s) = E_t[a(t + s)]$ and where $\hat{K}(\cdot)$ denotes the path of K given that a is certain to equal its expected value.

Now consider the situation where $a(t + \tau)$ is uncertain. Then the expected value, as of time t , of q at some future date $t + \tau$ can be written as the value of q in period t plus the "sum" of all the expected changes in q from time t to time $t + \tau$. More formally:

$$(3) E_t[q(t + \tau)] = q(t) + \int_{s=t}^{\tau} E_t[\dot{q}(t + s)] ds$$

Since equation (8.28) in the text -- $E_t[\dot{q}(t)] = r q(t) - \pi(K(t)) = r q(t) - a + bK(t)$ -- holds in all periods and since $\pi(K(t)) = a - bK(t)$, we can write:

$$(4) E_{t+s} [\dot{q}(t+s)] = r q(t+s) - a(t+s) + bK(t+s)$$

Taking the expected value, as of information available at time t , of both sides of equation (4) yields:

$$(5) E_t [\dot{q}(t+s)] = r E_t [q(t+s)] - E_t [a(t+s)] + b E_t [K(t+s)]$$

where we have used the law of iterated projections so that $E_t E_{t+s} [\dot{q}(t+s)] = E_t [\dot{q}(t+s)]$. Substituting equation (5) into equation (3) gives us:

$$(6) E_t [q(t+\tau)] = q(t) + \int_t^{\tau} [r E_t [q(t+s)] - E_t [a(t+s)] + b E_t [K(t+s)]] ds$$

If $E_t [q(t+\tau)] = \hat{q}(t+\tau, t)$ for all $\tau \geq 0$, then the right-hand sides of equations (2) and (6) must be equal for all $\tau \geq 0$. Thus:

$$(7) q(t) + \int_t^{\tau} [r \hat{q}(t+s, t) - E_t [a(t+s)] + b \hat{K}(t+s, t)] ds = \\ q(t) + \int_t^{\tau} [r E_t [q(t+s)] - E_t [a(t+s)] + b E_t [K(t+s)]] ds$$

Again, using $E_t [q(t+\tau)] = \hat{q}(t+\tau, t)$ for all $\tau \geq 0$, this simplifies to:

$$(8) b \int_t^{\tau} \hat{K}(t+s, t) ds = b \int_t^{\tau} E_t [K(t+s)] ds$$

Canceling the b 's and using Leibniz's rule to take the derivative of both sides of equation (8) with respect to τ gives us:

$$(9) \hat{K}(t+\tau, t) = E_t [K(t+\tau)]$$

Equation (9) holds for all $\tau \geq 0$.

b) Consider the situation where $a(t+\tau)$ is certain to equal $E_t [a(t+\tau)]$ for all $\tau \geq 0$ so that there is no uncertainty. Then from equation (8.22) in the text, we can write the market value of capital at time t as the present value of its future marginal revenue products:

$$(10) \hat{q}(t, t) = \int_{\tau=0}^{\infty} e^{-r\tau} [E_t [a(t+\tau)] - b \hat{K}(t+\tau, t)] d\tau$$

where we have used the facts that $\pi(K) = a - bK$ and $a(t+\tau) = E_t [a(t+\tau)]$. $\hat{q}(t, t)$ denotes the value of q given that a always equals its expected value, while $\hat{K}(t+\tau, t)$ is the same as in part (a) — it is the path of K given that a is always certain to equal its expected value.

Now consider the situation where $a(t+\tau)$ is uncertain. Then, using $\pi(K) = a - bK$, equation (8.26) in the text becomes:

$$(11) q(t) = \int_{\tau=0}^{\infty} e^{-r\tau} [E_t [a(t+\tau)] - b E_t [K(t+\tau)]] d\tau$$

As shown in part (a), if $E_t [q(t+\tau)] = \hat{q}(t+\tau, t)$ for all $\tau \geq 0$, then $E_t [K(t+\tau)] = \hat{K}(t+\tau, t)$ for all $\tau \geq 0$. This means that the right-hand sides of equations (10) and (11) are equal and so $q(t) = \hat{q}(t, t)$. That is, under these special circumstances — π is linear and the uncertainty concerns the intercept of the π function — the market value of capital is the same with the uncertainty as it is if the future value of the π function are certain to equal their expected values.

Even with uncertainty, each firm invests to the point where the cost of acquiring a unit of new capital equals the market value of capital. That is, investment satisfies:

$$(12) 1 + C'(I(t)) = q(t)$$

Since $C = \alpha I^2 / 2$, $C'(I) = \alpha I$. In addition, as we have just shown, $q(t) = \hat{q}(t, t)$. Thus equation (12) can be rewritten as:

$$(13) \quad 1 + \alpha I(t) = \hat{q}(t, t)$$

By definition, the change in each firm's capital stock is equal to $I(t)$. Since each firm faces the same $\hat{q}(t, t)$, they choose the same level of investment. Thus the change in the aggregate capital stock is given by $\dot{K}(t) = NI(t)$, where N is the number of firms. Substituting this expression into equation (13) yields:

$$(14) \quad 1 + \alpha \dot{K}(t)/N = \hat{q}(t, t)$$

Solving (14) for $\dot{K}(t)$ gives us:

$$(15) \quad \dot{K}(t) = N[\hat{q}(t, t) - 1]/\alpha$$

Under these special circumstances — π is linear, the uncertainty concerns the intercept of the π function and adjustment costs are quadratic — investment is the same with the uncertainty as it is if the future values of the π function are certain to be equal to their expected values.

Problem 8.11

a) If the firm does not undertake the investment, its expected profits are zero. Thus we have:

$$(1) \quad E(\pi^{\text{NO}}) = 0$$

If the firm does undertake the investment, its expected profits are the certain payoff in period 1 plus the expected payoff in period 2 less the cost of undertaking the investment. Thus we have:

$$(2) \quad E(\pi^{\text{YES}}) = \pi_1 + E(\pi_2) - I$$

The firm will undertake the investment if its expected profits from doing so are greater than its expected profits from not investing, or when:

$$(3) \quad E(\pi^{\text{YES}}) > E(\pi^{\text{NO}})$$

or simply when:

$$(4) \quad \pi_1 + E(\pi_2) - I > 0$$

b) Suppose the firm does not invest in period 1. Then in period 2, if $\pi_2 > I$, it will invest and earn $\pi_2 - I$. If $\pi_2 < I$, it will not invest in period 2 and will earn zero. Thus the expected profits from not investing in period 1 are:

$$(5) \quad E(\pi^{\text{NO IN 1}}) = \text{Prob}(\pi_2 > I) E[\pi_2 - I \mid \pi_2 > I]$$

From equation (2), the expected profits from investing in period 1 are:

$$(6) \quad E(\pi^{\text{YES IN 1}}) = \pi_1 + E(\pi_2) - I$$

Thus the difference in the firm's expected profits between not investing in period 1 and investing in period 1 are:

$$(7) \quad E(\pi^{\text{NO IN 1}}) - E(\pi^{\text{YES IN 1}}) = \text{Prob}(\pi_2 > I) E[\pi_2 - I \mid \pi_2 > I] - (\pi_1 + E(\pi_2) - I)$$

Even if $\pi_1 + E(\pi_2) - I > 0$, as long as $\text{Prob}(\pi_2 > I) E[\pi_2 - I \mid \pi_2 > I]$ is greater than $\pi_1 + E(\pi_2) - I$, the firm's expected profits are higher if it does not invest in period 1 than if it does.

c) The cost of waiting is that the firm foregoes any payoff in period 1. That is, it foregoes π_1 and hence:

$$(8) \quad \text{cost of waiting} = \pi_1$$

The benefit of waiting is that the firm can observe π_2 , see if it is less than I and decide not to invest and avoid a loss if this is the case. The expected loss that the firm avoids by waiting is equal to the probability that π_2 is less than I , multiplied by the expected loss given that $\pi_2 < I$, which is $E[I - \pi_2 \mid \pi_2 < I]$. Hence:

$$(9) \quad \text{benefit of waiting} = \text{Prob}(\pi_2 < I) E[I - \pi_2 \mid \pi_2 < I]$$

Note that by the definition of conditional expected values, we can write:

$$(10) \quad E(\pi_2 - I) = \text{Prob}(\pi_2 > I) E[\pi_2 - I \mid \pi_2 > I] + \text{Prob}(\pi_2 < I) E[\pi_2 - I \mid \pi_2 < I]$$

Substituting this into equation (7) yields:

$$(11) E(\pi^{\text{NO IN}}) - E(\pi^{\text{YES IN}}) = \text{Prob}(\pi_2 > I) E[\pi_2 - I | \pi_2 > I] - \pi_1 - \text{Prob}(\pi_2 > I) E[\pi_2 - I | \pi_2 > I] - \text{Prob}(\pi_2 < I) E[\pi_2 - I | \pi_2 < I]$$

Note that we can write $\text{Prob}(\pi_2 < I) E[\pi_2 - I | \pi_2 < I] = -\text{Prob}(\pi_2 < I) E[I - \pi_2 | \pi_2 < I]$. Using this fact, equation (11) becomes:

$$(12) E(\pi^{\text{NO IN}}) - E(\pi^{\text{YES IN}}) = -\pi_1 + \text{Prob}(\pi_2 < I) E[I - \pi_2 | \pi_2 < I]$$

Since π_1 is the cost of waiting and $\text{Prob}(\pi_2 < I) E[I - \pi_2 | \pi_2 < I]$ is the benefit of waiting, we do have:

$$(13) E(\pi^{\text{NO IN}}) - E(\pi^{\text{YES IN}}) = \text{benefit of waiting} - \text{cost of waiting}$$

Problem 8.12

a) Consider the value of a unit of debt. It pays off one unit of output at time $t + \tau$, for all $\tau \geq 0$. The consumer values this payoff according to the marginal utility of consumption at each time $t + \tau$. Thus the value of having one unit of output at time $t + \tau$ rather than at t is equal to the discounted marginal utility of consumption at time $t + \tau$ relative to the marginal utility of consumption at time t , which is given by $e^{-\rho\tau} u'(C(t + \tau)) / u'(C(t))$. Thus the value of a unit of debt at time t is simply the appropriately discounted "sum" of all the future payoffs, or:

$$(1) P(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t + \tau))}{u'(C(t))} \right] d\tau$$

Equity holders are the residual claimant and thus at time $t + \tau$, $\tau \geq 0$, they receive the additional profit generated by the marginal unit of capital, $\pi(K(t + \tau))$, minus the total amount paid to bond holders, which is b (the total number of outstanding bonds). Again, individuals value this payoff at time $t + \tau$ according to the discounted marginal utility of consumption at time $t + \tau$ relative to the marginal utility of consumption at time t . Thus the value of the equity in the marginal unit of capital is:

$$(2) V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t + \tau))}{u'(C(t))} (\pi(K(t + \tau)) - b) \right] d\tau$$

b) Adding equation (2) to b times equation (1) gives us the market value of the claim on the marginal unit of capital:

$$(3) P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} b E_t \left[\frac{u'(C(t + \tau))}{u'(C(t))} \right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t + \tau))}{u'(C(t))} (\pi(K(t + \tau)) - b) \right] d\tau$$

Combining the integrals yields:

$$(4) P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t + \tau))}{u'(C(t))} (b + \pi(K(t + \tau)) - b) \right] d\tau$$

and thus:

$$(5) P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t + \tau))}{u'(C(t))} \pi(K(t + \tau)) \right] d\tau$$

The division of financing between bonds and equity — as captured by b , the number of outstanding bonds — does not affect the market value of the claims on the marginal unit of capital. The present discounted value of that unit of capital is determined by its expected effect on the path of profits. Since the division of $\pi(K(t + \tau))$ between bonds and equity does not affect the size of $\pi(K(t + \tau))$, it does not affect the market value of the claim on the unit of capital.

c) The market value of each of the n assets is given by:

$$(6) \quad V_i(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t+\tau))}{u'(C(t))} d_i(t+\tau) \right] d\tau$$

There will be n equations of the form of (6). Adding these n equations together gives us the total value of the n financial instruments:

$$(7) \quad V_1(t) + \dots + V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t+\tau))}{u'(C(t))} (d_1(t+\tau) + \dots + d_n(t+\tau)) \right] d\tau$$

Since $d_1(t+\tau) + \dots + d_n(t+\tau) = \pi(K(t+\tau))$, we can rewrite equation (7) as:

$$(8) \quad V_1(t) + \dots + V_n(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t+\tau))}{u'(C(t))} \pi(K(t+\tau)) \right] d\tau$$

The total market value of the n financial instruments is determined by the expected effect on the path of profits of the marginal unit of capital. It does not depend upon the individual payoffs to the assets.

d) The value of a unit of debt continues to be given by equation (1). The value of a unit of equity is now:

$$(9) \quad V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t+\tau))}{u'(C(t))} \{ (1-\theta) [\pi(K(t+\tau)) - b] \} \right] d\tau$$

Adding equation (9) to b times equation (1) gives the market value of the claims on the marginal unit of capital:

$$P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} b E_t \left[\frac{u'(C(t+\tau))}{u'(C(t))} \right] d\tau + \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t+\tau))}{u'(C(t))} \{ (1-\theta) [\pi(K(t+\tau)) - b] \} \right] d\tau$$

Combining the integrals yields:

$$(10) \quad P(t)b + V(t) = \int_{\tau=0}^{\infty} e^{-\rho\tau} E_t \left[\frac{u'(C(t+\tau))}{u'(C(t))} [(1-\theta)\pi(K(t+\tau)) + \theta b] \right] d\tau$$

Now the division of the financing between bonds and equity does matter. The number of bonds issued, b , does affect the market value of the claim on the marginal unit of capital. The division of the additional profits between bonds and capital does affect the size of those profits. Specifically, a switch toward debt financing increases profits since interest payments are tax deductible.

SOLUTIONS TO CHAPTER 9

Problem 9.1

a) From $m_t - p_t = c - b(E_t p_{t+1} - p_t)$, collecting the terms in p_t yields:

$$p_t(1+b) = m_t - c + bE_t p_{t+1}$$

and so p_t is given by:

$$(1) \quad p_t = \left(\frac{b}{1+b}\right) E_t p_{t+1} + \left(\frac{1}{1+b}\right) \cdot (m_t - c)$$

b) Equation (1) holds in all periods so that we can write p_{t+1} as:

$$(2) \quad p_{t+1} = \left(\frac{b}{1+b}\right) E_{t+1} p_{t+2} + \left(\frac{1}{1+b}\right) \cdot (m_{t+1} - c)$$

Taking the expected value, as of time t , of both sides of equation (2) yields:

$$(3) \quad E_t p_{t+1} = \left(\frac{b}{1+b}\right) E_t p_{t+2} + \left(\frac{1}{1+b}\right) \cdot (E_t m_{t+1} - c)$$

where we have used the law of iterated projections, which states that $E_t E_{t+1} p_{t+2} = E_t p_{t+2}$. If this did not hold, individuals would be expecting to revise their estimate of p_{t+2} either up or down, which would imply that their original estimate was not rational.

c) Substituting equation (3) into equation (1) yields:

$$(4) \quad p_t = \left(\frac{b}{1+b}\right)^2 E_t p_{t+2} + \left(\frac{1}{1+b}\right) \cdot \left[(m_t - c) + \left(\frac{b}{1+b}\right) \cdot (E_t m_{t+1} - c) \right]$$

Again using the fact that equation (1) holds in all periods, we can write p_{t+2} as:

$$(5) \quad p_{t+2} = \left(\frac{b}{1+b}\right) E_{t+2} p_{t+3} + \left(\frac{1}{1+b}\right) \cdot (m_{t+2} - c)$$

Taking the expected value, as of time t , of both sides of equation (5) gives us:

$$(6) \quad E_t p_{t+2} = \left(\frac{b}{1+b}\right) E_t p_{t+3} + \left(\frac{1}{1+b}\right) \cdot (E_t m_{t+2} - c)$$

where we have again used the law of iterated projections so that $E_t E_{t+2} p_{t+3} = E_t p_{t+3}$. Substituting equation (6) into equation (4) leaves us with:

$$(7) \quad p_t = \left(\frac{b}{1+b}\right)^3 E_t p_{t+3} + \left(\frac{1}{1+b}\right) \cdot \left[(m_t - c) + \left(\frac{b}{1+b}\right) \cdot (E_t m_{t+1} - c) + \left(\frac{b}{1+b}\right)^2 \cdot (E_t m_{t+2} - c) \right]$$

The pattern should now be clear. We can write p_t as an infinite sum:

$$(8) \quad p_t = \left(\frac{1}{1+b}\right) \cdot \left[(m_t - c) + \left(\frac{b}{1+b}\right) (E_t m_{t+1} - c) + \left(\frac{b}{1+b}\right)^2 (E_t m_{t+2} - c) + \left(\frac{b}{1+b}\right)^3 (E_t m_{t+3} - c) + \dots \right]$$

d) With output and the real interest rate constant, the price level must adjust to clear the money market. If m_{t+i} is higher, p_{t+i} will need to be higher to clear the money market. Thus if people expect, in period $t+i-1$, that m_{t+i} will be higher they will also expect p_{t+i} to be higher. Thus in period $t+i-1$, expected inflation will be higher. This reduces real money demand in period $t+i-1$. For a given value of m_{t+i-1} , this means that p_{t+i-1} will need to rise to clear the money market. Now go back one more period. Suppose that people expect, in period $t+i-2$, that m_{t+i} will be higher. Then they expect -- through the reasoning above -- that

p_{t+i-1} will be higher. Thus expected inflation in $t+i-2$ will be higher, real money demand will be lower and thus p_{t+i-2} will be need to be higher to clear the money market. Reasoning backward, as soon as people expect the nominal money supply to rise in some future period, the price level will rise in the current period.

e) Equation (8) can be written using summation notation as:

$$(9) \quad p_t = \frac{1}{1+b} \cdot \sum_{i=0}^{\infty} \left(\frac{b}{1+b} \right)^i (E_t m_{t+i} - c)$$

Substituting the assumption that $E_t m_{t+i} = m_t + gi$ into equation (9) yields:

$$(10) \quad p_t = \frac{1}{1+b} \cdot \sum_{i=0}^{\infty} \left(\frac{b}{1+b} \right)^i (m_t + gi - c) = \frac{1}{1+b} \left[(m_t - c) \sum_{i=0}^{\infty} \left(\frac{b}{1+b} \right)^i + g \sum_{i=0}^{\infty} i \left(\frac{b}{1+b} \right)^i \right]$$

Now we can use the facts that:

$$(11) \quad \sum_{i=0}^{\infty} \left(\frac{b}{1+b} \right)^i = 1 + \left(\frac{b}{1+b} \right) + \left(\frac{b}{1+b} \right)^2 + \dots = \frac{1}{1 - [b/(1+b)]} = 1+b$$

and:

$$(12) \quad \sum_{i=0}^{\infty} i \left(\frac{b}{1+b} \right)^i = \frac{b/(1+b)}{\{1 - [b/(1+b)]\}^2} = \frac{b/(1+b)}{1/(1+b)^2} = b(1+b)$$

Equation (12) uses the result that for some x :

$$(13) \quad \sum_{i=0}^{\infty} i x^i = \frac{x}{(1-x)^2}$$

A (not entirely rigorous) way to see why (13) and thus (12) hold is to note that with $x < 1$:

$$(14) \quad 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

Differentiating both sides of equation (14) with respect to x (which means differentiating term by term on the left-hand side) gives us:

$$(15) \quad 1 + 2x + 3x^2 + \dots = \frac{1}{(1-x)^2}$$

Multiplying both sides of equation (15) by x yields:

$$(16) \quad x + 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2}$$

Note that (16) and (13) are equivalent — the left-hand side of equation (13) is simply the left-hand side of (16) written in summation notation.

Substituting equations (11) and (12) into equation (10) yields:

$$(17) \quad p_t = \frac{1}{1+b} [(m_t - c)(1+b) + gb(1+b)]$$

Thus, the price level is given by:

$$(18) \quad p_t = (m_t - c) + bg$$

To see how the price level changes when money growth changes, use equation (18) to take the derivative of p_t with respect to g :

$$(19) \quad \partial p_t / \partial g = b > 0$$

Thus a rise in money growth — even without a rise in the level of the current period's money supply — causes an upward jump in the current price level.

Problem 9.2

a) Substituting the normalized, flexible-price level of output, $y_0 = 0$, into the IS equation — $y_0 = c - ar_0$ gives us $0 = c - ar_0$. Solving for the real interest rate in period 0 yields:

$$(1) \quad r_0 = c/a$$

Since the nominal money stock is expected to be constant, the price level is expected to be constant and thus expected inflation from period 0 to period 1 is:

$$(2) \quad E_0(p_1) - p_0 = 0$$

The nominal interest rate in period 0 — $i_0 = r_0 + [E_0(p_1) - p_0]$ — is simply equal to the real interest rate:

$$(3) \quad i_0 = c/a$$

Finally, substituting the assumptions that $m_0 = 0$ and $y_0 = 0$ as well as equation (3) into the LM curve — $m_0 - p_0 = b + hy_0 - ki_0$ — yields $-p_0 = b - (ck/a)$ or simply:

$$(4) \quad p_0 = (ck/a) - b$$

b) In period 2, the economy is once again at the flexible-price equilibrium level of output, which is 0. Substituting this fact into the IS equation allows us to solve for the real interest rate in period 2:

$$(5) \quad r_2 = c/a$$

Since expected inflation from period 2 to period 3 is equal to g — the price level is expected to rise by the same amount as the nominal money supply each period — the nominal interest rate in period 2 is given by:

$$(6) \quad i_2 = (c/a) + g$$

Since m was equal to 0 in period 0 and then increases by g in each following period, the nominal money supply in period 2 is $m_2 = 2g$. Substituting this fact as well as $y_2 = 0$ and $i_2 = (c/a) + g$ into the LM equation leaves us with:

$$2g - p_2 = b - (ck/a) - kg$$

Solving for p_2 gives us:

$$(7) \quad p_2 = -b + (ck/a) + (2 + k)g$$

c) The price level is completely unresponsive to unanticipated monetary shocks for one period. Thus the price level in period 1 does not change from its period 0 value and hence:

$$(8) \quad p_1 = (ck/a) - b$$

The expectation of inflation from period 1 to period 2, $E_1(p_2) - p_1$, is therefore:

$$(9) \quad E_1(p_2) - p_1 = -b + (ck/a) + (2 + k)g - (ck/a) + b = (2 + k)g$$

where we have used equations (7) and (8) to substitute for p_2 and p_1 .

Now substitute the IS equation — $y_1 = c - ar_1$ — into the LM equation — $m_1 - p_1 = b + hy_1 - ki_1$ — to obtain:

$$(10) \quad m_1 - p_1 = b + hc - ar_1 - ki_1$$

By assumption, the nominal money supply in period 1 is g . In addition, $i_1 = r_1 + [E_1(p_2) - p_1]$, which — using equation (9) — is equivalent to $i_1 = r_1 + (2 + k)g$. Substituting these facts as well as equation (8) for the price level into equation (10) gives us:

$$(11) \quad g - (ck/a) + b = b + hc - ar_1 - k[r_1 + (2 + k)g]$$

Simplifying and collecting the terms in r_1 yields:

$$(12) \quad r_1[ah + k] = hc + (ck/a) - g - (2 + k)kg$$

Thus the real interest rate in period 1 is given by:

$$(13) \quad r_1 = \frac{hc + (ck/a) - g - (2 + k)kg}{ah + k}$$

Finally, substituting equations (9) and (13) into $i_1 = r_1 + [E_1(p_2) - p_1]$ gives us:

$$(14) \quad i_1 = \frac{hc + (ck/a) - g - (2 + k)kg}{ah + k} + (2 + k)g = \frac{hc + (ck/a) - g - (2 + k)kg + (2 + k)ahg + (2 + k)kg}{ah + k}$$

Thus the nominal interest rate in period 1 is given by:

$$(15) \quad i_1 = \frac{hc + (ck/a) - g + (2+k)ahg}{ah + k}$$

d) Using equations (15) and (3), the change in the nominal interest rate from period 0 to period 1 is:

$$i_1 - i_0 = \frac{hc + (ck/a) - g + (2+k)ahg}{ah + k} - (c/a) = \frac{hc + (ck/a) - g + (2+k)ahg - hc - (ck/a)}{ah + k}$$

Simplifying yields:

$$(16) \quad i_1 - i_0 = \frac{(2+k)ahg - g}{ah + k}$$

We can determine the condition required of the parameters in order for the nominal interest rate to fall from period 0 to period 1 — that is, for $i_1 - i_0 < 0$. From equation (16), this condition is:

$$\frac{g[(2+k)ah - 1]}{ah + k} < 0$$

or simply:

$$(17) \quad (2+k)ah < 1$$

The smaller is a (the elasticity of output with respect to changes in the real interest rate), the smaller is h (the income elasticity of real money demand) and the smaller is k (the interest semi-elasticity of real money demand), the more likely it is for the condition in (17) to be satisfied and thus the more likely it is for the nominal interest rate to fall in response to the monetary expansion.

For the nominal interest rate, $i = r + \pi^e$, to fall, we need the liquidity effect to outweigh the expected inflation effect. That is, we need the real interest rate to fall by more than expected inflation rises. With the price level fixed by assumption in period 1, y and i must adjust to ensure money market equilibrium. If k is small, changes in i will not affect real money demand very much. We need y to rise to increase real money demand and get it equal to the new higher real money stock. If h is small, we need y to rise a lot in order to accomplish this. If y is to rise a lot, we need — from the IS equation — the real interest rate to fall a lot. If furthermore, a is small, we need r to fall a lot just to generate an increase in output. Thus small values of k , h and a all work to make the drop in r larger and thus make it more likely that i will fall.

Problem 9.3

a) Any shock to the money supply in period $t+1$ is fully reflected in the price level by period $t+2$. That is, the only reason the price level will change from period $t+1$ to period $t+2$ is if there is a non-zero realization of u in period $t+1$. From the law of iterated projections, we have:

$$(1) \quad E_t [E_{t+1} (p_{t+2}) - p_{t+1}] = E_t (p_{t+2} - p_{t+1})$$

Since the expected value, as of period t , of u_{t+1} is zero, the price level is not expected to change from period $t+1$ to period $t+2$. Thus:

$$(2) \quad E_t [E_{t+1} (p_{t+2}) - p_{t+1}] = 0$$

Since the LM equation must hold each period, we can write:

$$(3) \quad m_{t+1} - p_{t+1} = b + hy_{t+1} - kr_{t+1} - k[E_{t+1} (p_{t+2}) - p_{t+1}]$$

where we have substituted in for $i_{t+1} = r_{t+1} + [E_{t+1} (p_{t+2}) - p_{t+1}]$. Taking the expected value of both sides of equation (3) yields:

$$(4) \quad E_t m_{t+1} - E_t p_{t+1} = b + h\bar{y} - k\bar{r}$$

where we have used the result from equation (2) that $E_t [E_{t+1} (p_{t+2}) - p_{t+1}] = 0$. In addition, since y_{t+1} and r_{t+1} will only depend on the u_{t+1} shock, which is expected to be zero, they are expected to be equal to their average values.

b) Rearranging equation (4), we have:

$$(5) E_t p_{t+1} = E_t m_{t+1} - b - h\bar{y} + k\bar{r}$$

Since $m_{t+1} = m_t + u_{t+1}$, $E_t m_{t+1} = m_t$. Using this fact and subtracting p_t from both sides of equation (5) yields:

$$(6) E_t p_{t+1} - p_t = (m_t - p_t) - b - h\bar{y} + k\bar{r}$$

As explained in part (a), expected inflation is equal to u_t and thus we can write:

$$(7) u_t = (m_t - p_t) - b - h\bar{y} + k\bar{r}$$

Substituting $m_t = m_{t-1} + u_t$ into equation (7) and rearranging to solve for p_t yields:

$$(8) p_t = m_{t-1} - b - h\bar{y} + k\bar{r}$$

The next step is to solve for output in period t . Rearranging the LM equation to solve for i_t yields:

$$(9) i_t = [b + hy_t - (m_t - p_t)]/k$$

From equation (7):

$$(10) (m_t - p_t) = u_t + b + h\bar{y} - k\bar{r}$$

Substituting equation (10) into equation (9) gives us:

$$(11) i_t = \frac{b + hy_t - u_t - b - h\bar{y} + k\bar{r}}{k} = \frac{h(y_t - \bar{y}) + k\bar{r} - u_t}{k}$$

Substituting equation (11) for i_t and using the fact that $\pi_t^e = u_t$, the IS equation becomes:

$$(12) y_t = c - a \left[\frac{h(y_t - \bar{y}) + k\bar{r} - u_t}{k} \right] + au_t$$

Collecting the terms in y_t , we have:

$$(13) \left[\frac{k + ah}{k} \right] y_t = c + \frac{ah\bar{y} - ak\bar{r} + au_t}{k} + au_t$$

which implies:

$$(14) y_t = \frac{kc + ah\bar{y} - ak\bar{r} + au_t + kau_t}{k + ah}$$

and thus output in period t is given by:

$$(15) y_t = \frac{kc + a[h\bar{y} - k\bar{r} + (1+k)u_t]}{k + ah}$$

In order to determine the real interest rate, rearrange the IS equation to obtain:

$$(16) r_t = (c/a) - (y_t/a)$$

Substituting equation (15) into equation (16) yields:

$$(17) r_t = \frac{c}{a} - \frac{kc + a[h\bar{y} - k\bar{r} + (1+k)u_t]}{a(k + ah)}$$

which implies:

$$(18) r_t = \frac{ck + cah - kc - a[h\bar{y} - k\bar{r} + (1+k)u_t]}{a(k + ah)} = \frac{ch - [h\bar{y} - k\bar{r} + (1+k)u_t]}{k + ah}$$

Thus the real interest rate in period t is:

$$(19) r_t = \frac{h(c - \bar{y}) + k\bar{r} - (1+k)u_t}{k + ah}$$

The nominal interest rate is $i_t = r_t + \pi_t^e$, where $\pi_t^e = u_t$:

$$(20) i_t = r_t + u_t$$

Substituting equation (19) into equation (20) gives us:

$$(21) \quad i_t = \frac{h(c - \bar{y}) + k\bar{r} - (1+k)u_t + (k+ah)u_t}{k+ah} = \frac{h(c - \bar{y}) + k\bar{r} + (ah-1)u_t}{k+ah}$$

c) From equation (21), with $\pi_t^e = u_t$, we have:

$$(22) \quad i_t = \frac{h(c - \bar{y}) + k\bar{r}}{k+ah} + \frac{(ah-1)}{k+ah} \pi_t^e$$

From equation (22), we can see that changes in expected inflation are not reflected one-for-one in the nominal rate. This is due to the fact that prices are completely unresponsive to the monetary disturbance for one period. This means that, in general, output and the nominal interest rate will adjust to clear the money market. In order for output to change, the real interest must change and therefore, in general, the nominal interest rate will not move one-for-one with inflation.

Problem 9.4

a) Under rational expectations:

$$(1) \quad \pi_{t+1} = E_t \pi_{t+1} + \varepsilon_{t+1}$$

where ε_{t+1} is a disturbance that is uncorrelated with anything known at t . Now consider the regression:

$$(2) \quad i_t = a + b\pi_{t+1} + e_t$$

Using the hint in the question, the OLS estimator of b is given by:

$$(3) \quad \hat{b} = \frac{\text{cov}(i_t, \pi_{t+1})}{\text{var}(\pi_{t+1})}$$

Using $i_t = r + E_t \pi_{t+1}$ and equation (1), we can write the covariance in the numerator as:

$$(4) \quad \text{cov}(i_t, \pi_{t+1}) = \text{cov}(r + E_t \pi_{t+1}, E_t \pi_{t+1} + \varepsilon_{t+1})$$

Since r and $E_t \pi_{t+1}$ are uncorrelated and ε_{t+1} is uncorrelated with anything known at t , this implies:

$$(5) \quad \text{cov}(i_t, \pi_{t+1}) = \text{var}(E_t \pi_{t+1})$$

Again using equation (1), the variance in the denominator of equation (3) can be written as:

$$(6) \quad \text{var}(\pi_{t+1}) = \text{var}(E_t \pi_{t+1} + \varepsilon_{t+1}) = \text{var}(E_t \pi_{t+1}) + \text{var}(\varepsilon_{t+1})$$

where we have used the fact that $\text{cov}(E_t \pi_{t+1}, \varepsilon_{t+1}) = 0$.

Substituting equations (5) and (6) into equation (3) allows us to write the OLS estimator as:

$$(7) \quad \hat{b} = \frac{\text{var}(E_t \pi_{t+1})}{\text{var}(E_t \pi_{t+1}) + \text{var}(\varepsilon_{t+1})} < 1$$

The hypothesis that the real interest rate is constant – so that changes in expected inflation cause one-for-one movements in the nominal interest rate – only predicts that the coefficient on π_{t+1} should be positive and less than one – not that it will take on any specific value.

b) Now consider a regression of the form:

$$(8) \quad \pi_{t+1} = a' + b' i_t + e_t'$$

The OLS estimator of b' is of the form:

$$(9) \quad \hat{b}' = \frac{\text{cov}(i_t, \pi_{t+1})}{\text{var}(i_t)}$$

The covariance in the numerator of equation (9) is still given by equation (5). Since $i_t = r + E_t \pi_{t+1}$, we can write the denominator of equation (9) as:

$$(10) \quad \text{var}(i_t) = \text{var}(r) + \text{var}(E_t \pi_{t+1})$$

where we have used the fact that $\text{cov}(r, E_t \pi_{t+1}) = 0$.

Substituting equations (5) and (10) into equation (9) gives us the OLS estimator:

$$(11) \hat{b}' = \frac{\text{var}(E_t \pi_{t+1})}{\text{var}(r) + \text{var}(E_t \pi_{t+1})}$$

The hypothesis that the real interest rate is constant -- so that $\text{var}(r) = 0$ -- predicts a coefficient of 1 on i_t .

c) Consider the regression:

$$(12) i_t = a + b_0 \pi_t + b_1 \pi_{t-1} + \dots + b_n \pi_{t-n} + \varepsilon_t$$

So, for example, the coefficient b_0 represents the direct impact on i_t of a change in π_t , holding the other π 's constant.

Now suppose that the behavior of actual inflation is given by:

$$(13) \pi_t = \rho \pi_{t-1} + e_t$$

If $i_t = r + E_t \pi_{t+1}$, with r constant, changes in expected inflation should cause one-for-one movements in i_t . Thus since $\pi_{t+1} = \rho \pi_t + e_{t+1}$, a change in π_t of $\Delta \pi_t$ will cause $E_t \pi_{t+1}$ -- and thus i_t -- to change by $\rho \Delta \pi_t$. So we would expect $b_0 = \rho$ in the above regression.

But now, controlling for π_t , the other inflation variables -- $\pi_{t-1}, \dots, \pi_{t-n}$ -- provide no new information about π_{t+1} . Any effect that π_{t-1} , say, has on π_{t+1} is already captured indirectly by π_{t-1} 's impact on π_t . Thus we would expect $b_1 = \dots = b_n = 0$ in the above regression. Thus the claim is incorrect since we would have $b_0 + b_1 + \dots + b_n = \rho$, not $b_0 + b_1 + \dots + b_n = 1$.

Problem 9.5

a) We have $\pi_t = p_t - p_{t-1}$ and $\pi_t^e = p_t^e - p_{t-1}$. Thus $\pi_t - \pi_t^e = (p_t - p_{t-1}) - (p_t^e - p_{t-1}) = p_t - p_t^e$. We can therefore write the Lucas supply function as:

$$(1) y_t = \bar{y} + b(p_t - p_t^e)$$

Setting aggregate supply equal to aggregate demand (which is given by $y_t = m_t - p_t$) gives us:

$$(2) m_t - p_t = \bar{y} + b(p_t - p_t^e)$$

Solving equation (2) for p_t yields:

$$(3) p_t = \frac{1}{1+b} m_t + \frac{b}{1+b} p_t^e - \frac{1}{1+b} \bar{y}$$

With rational expectations, the expected value of both sides of equation (3) must be equal. Hence:

$$(4) p_t^e = \frac{1}{1+b} (m_{t-1} + a) + \frac{b}{1+b} p_t^e - \frac{1}{1+b} \bar{y}$$

where we have used the fact that the expected value of $m_t = m_{t-1} + a + \varepsilon_t$ is equal to $m_{t-1} + a$ since ε is white noise.

Subtracting equation (4) from equation (3) yields:

$$(5) p_t - p_t^e = \frac{1}{1+b} m_t - \frac{1}{1+b} (m_{t-1} + a) = \frac{1}{1+b} (m_t - m_{t-1} - a)$$

Substituting (5) into equation (1) gives us:

$$(6) y_t = \bar{y} + \frac{b}{1+b} (m_t - m_{t-1} - a)$$

b) From equation (6), we can see that we also need to know a , as well as m_t and m_{t-1} , in order to determine the current level of output. Intuitively, equation (6) says that only unexpected money affects output since the difference between m_t and $(m_{t-1} + a)$ is the random shock, ε_t . However, if we don't know a , we cannot determine how much of the change in the nominal money stock from period $t-1$ to period t was due to a (and thus was expected) and how much was due to ε (and thus was unexpected).

c) Again, it must be true that with rational expectations, the expected value of both sides of equation (3) must be equal. However, the expected value of m_t is now $m_{t-1} + \rho(0) + (1 - \rho)a = m_{t-1} + (1 - \rho)a$ since private agents believe that the probability that $a = 0$ is ρ . Thus:

$$(7) \quad p_t^e = \frac{1}{1+b} [m_{t-1} + (1-\rho)a] + \frac{b}{1+b} p_t^e - \frac{1}{1+b} \bar{y}$$

Subtracting equation (7) from equation (3) yields:

$$(8) \quad p_t - p_t^e = \frac{1}{1+b} [m_t - m_{t-1} - (1-\rho)a]$$

Substituting equation (8) into equation (1) gives us:

$$(9) \quad y_t = \bar{y} + \frac{b}{1+b} [m_t - m_{t-1} - (1-\rho)a]$$

d) Equation (6) holds in any period in which there is no regime shift. Thus if there is no regime shift in period $t-1$, we can write:

$$(10) \quad y_{t-1} = \bar{y} + \frac{b}{1+b} (m_{t-1} - m_{t-2} - a)$$

Subtracting equation (10) from equation (6) yields:

$$(11) \quad y_t - y_{t-1} = \frac{b}{1+b} [(m_t - m_{t-1}) - (m_{t-1} - m_{t-2})]$$

Defining $\Delta y_t \equiv y_t - y_{t-1}$ and $\Delta m_t \equiv m_t - m_{t-1}$, we have:

$$(12) \quad \Delta y_t = \frac{b}{1+b} [\Delta m_t - \Delta m_{t-1}]$$

Equation (12) states that in the absence of regime shifts, output growth is determined by the change in money growth.

If there is a regime shift in period t , equation (9) holds. Subtracting equation (10) from equation (9) yields:

$$y_t - y_{t-1} = \frac{b}{1+b} [(m_t - m_{t-1}) - (m_{t-1} - m_{t-2})] + \frac{b}{1+b} [a - (1-\rho)a]$$

or simply:

$$(13) \quad \Delta y_t = \frac{\rho ab}{1+b} + \frac{b}{1+b} [\Delta m_t - \Delta m_{t-1}]$$

Under the null hypothesis of no credibility of the announcement of the regime shift, $\rho = 0$, the first term on the right-hand side of equation (13) is equal to zero. Thus if the announcement is not believed, equations (13) and (12) are identical. Thus we can run a regression of Δy_t on $[\Delta m_t - \Delta m_{t-1}]$ and a dummy variable that equals 1 in the period of a regime shift. The coefficient on that dummy variable will reflect the amount of credibility of the policymaker's announcement. In fact, since we will have an estimate of $b/(1+b)$ and can determine a (the average change in the money stock before the regime shift), we can calculate an estimate of ρ from the coefficient on the dummy variable.

Problem 9.6

a) i) The one-period nominal interest rate is given by $i_t^1 = E_t \pi_{t+1}$ since the real interest rate is assumed constant at zero. Since $\pi_{t+1} = \Delta m_{t+1}$, we have:

$$(1) \quad i_t^1 = E_t \Delta m_{t+1}$$

Since money growth is given by:

$$(2) \quad \Delta m_t = k \Delta m_{t-1} + \varepsilon_t$$

and since equation (2) holds in all periods, we can write:

$$(3) \quad \Delta m_{t-1} = k \Delta m_t + \varepsilon_{t-1}$$

Substituting equation (3) into equation (1), we have:

$$(4) \quad i_t^1 = E_t(k\Delta m_t + \varepsilon_{t+1}) = k\Delta m_t$$

where we have used the fact that Δm_t is known as of time t and $E_t(\varepsilon_{t+1}) = 0$.

a) ii) The expectation, as of time t , of the nominal interest rate from period $t + 1$ to $t + 2$ is:

$$(5) \quad E_t i_{t+1}^1 = E_t \pi_{t+2} = E_t \Delta m_{t+2}$$

Since equation (2) holds every period, we can write:

$$(6) \quad \Delta m_{t+2} = k\Delta m_{t+1} + \varepsilon_{t+2}$$

Substituting equation (2) into equation (6) gives us Δm_{t+2} as a function of Δm_t :

$$(7) \quad \Delta m_{t+2} = k^2 \Delta m_t + k\varepsilon_{t+1} + \varepsilon_{t+2}$$

Substituting equation (7) into equation (5) gives us:

$$(8) \quad E_t i_{t+1}^1 = E_t(k^2 \Delta m_t + k\varepsilon_{t+1} + \varepsilon_{t+2}) = k^2 \Delta m_t$$

where we have used the fact that Δm_t is known at t and the ε 's are mean-zero disturbances.

a) iii) Under the rational-expectations theory of the term structure, the two-period interest rate is given by:

$$(9) \quad i_t^2 = [i_t^1 + E_t i_{t+1}^1]/2$$

Substituting equation (8) into equation (9), we have:

$$(10) \quad i_t^2 = [i_t^1 + k^2 \Delta m_t]/2$$

From equation (4), $k\Delta m_t = i_t^1$ and so equation (10) can be rewritten as:

$$(11) \quad i_t^2 = [i_t^1 + ki_t^1]/2 = i_t^1(1+k)/2$$

a) iv) From equation (11), a rise in k will increase the two-period interest rate, i_t^2 , for any given one-period rate. For a given level of inflation in period t , expected inflation for period $t + 1$ will now be higher. Thus for a given one-period interest rate in t , the one-period rate in $t + 1$ is expected to be higher. Therefore i_t^2 , which is the average of the one-period rate in t and the expected one-period rate in $t + 1$, will now be higher for a given i_t^1 .

Note that as k goes to 1 – so that money growth and thus inflation approach a random walk, the two-period interest rate becomes equal to the one-period interest rate. That is because with inflation a random walk, next period's inflation (and thus next period's one-period nominal rate) is expected to be equal to this period's inflation (and thus this period's one-period nominal rate).

b) i) Equation (4) holds in all periods and thus the actual one-period interest rate in $t + 1$ is:

$$(12) \quad i_{t+1}^1 = k\Delta m_{t+1}$$

Substituting equation (3) into equation (12) yields:

$$(13) \quad i_{t+1}^1 = k^2 \Delta m_t + k\varepsilon_{t+1}$$

Thus:

$$(14) \quad i_{t+1}^1 - i_t^1 = k^2 \Delta m_t + k\varepsilon_{t+1} - k\Delta m_t = k(k-1)\Delta m_t + k\varepsilon_{t+1}$$

From equation (11), we can write:

$$(15) \quad i_t^2 - i_t^1 = [i_t^1(1+k)/2] - i_t^1 = [i_t^1(1+k-2)/2]$$

or substituting in for $i_t^1 = k\Delta m_t$, we have:

$$(16) \quad i_t^2 - i_t^1 = \frac{k(k-1)\Delta m_t}{2}$$

Using the hint in the question, the OLS estimator of b in the regression:

$$(17) \quad i_{t+1}^1 - i_t^1 = a + b[i_t^2 - i_t^1] + e_{t+1}$$

is given by:

$$(18) \hat{b} = \frac{\text{cov}[(i_{t+1}^1 - i_t^1), (i_t^2 - i_t^1)]}{\text{var}(i_t^2 - i_t^1)}$$

Using equations (14) and (16), the covariance in the numerator can be written as:

$$(19) \text{cov}[(i_{t+1}^1 - i_t^1), (i_t^2 - i_t^1)] = \text{cov}\left[k(k-1)\Delta m_t + k\varepsilon_{t+1}, \frac{k(k-1)\Delta m_t}{2}\right]$$

Since ε is white noise and $\text{var}(\Delta m_t) = \sigma_\varepsilon^2$, we have:

$$(20) \text{cov}[(i_{t+1}^1 - i_t^1), (i_t^2 - i_t^1)] = \frac{k^2(k-1)^2}{2} \sigma_\varepsilon^2$$

Using equation (16), the variance in the denominator of equation (18) can be written as:

$$(21) \text{var}(i_t^2 - i_t^1) = \frac{k^2(k-1)^2}{4} \sigma_\varepsilon^2$$

Substituting equations (20) and (21) into equation (18) gives us:

$$(22) \hat{b} = \frac{\frac{k^2(k-1)^2}{2} \sigma_\varepsilon^2}{\frac{k^2(k-1)^2}{4} \sigma_\varepsilon^2} = 2$$

b) ii) With the time-varying term premium, equation (16) becomes:

$$(23) i_t^2 - i_t^1 = \frac{k(k-1)\Delta m_t}{2} + \theta_t$$

Using equations (14) and (23), the covariance in the numerator of equation (18) is now given by:

$$(24) \text{cov}[(i_{t+1}^1 - i_t^1), (i_t^2 - i_t^1)] = \text{cov}\left[k(k-1)\Delta m_t + k\varepsilon_{t+1}, \frac{k(k-1)\Delta m_t}{2} + \theta_t\right]$$

Since ε and θ are white noise, this is simply:

$$(25) \text{cov}[(i_{t+1}^1 - i_t^1), (i_t^2 - i_t^1)] = \frac{k^2(k-1)^2}{2} \sigma_\varepsilon^2$$

This covariance is the same as it was without the time-varying term premium. However, the variance of $(i_t^2 - i_t^1)$ will change. It is now given by:

$$(26) \text{var}(i_t^2 - i_t^1) = \frac{k^2(k-1)^2}{4} \sigma_\varepsilon^2 + \sigma_\theta^2$$

where we have used the fact that the covariance between ε and θ is zero.

Substituting equations (25) and (26) into equation (18) gives us:

$$(27) \hat{b} = \frac{\frac{k^2(k-1)^2}{2} \sigma_\varepsilon^2}{\left(\frac{k^2(k-1)^2}{4} \sigma_\varepsilon^2 + \sigma_\theta^2\right)} = \frac{2}{1 + \left(\frac{4\sigma_\theta^2}{k^2(k-1)^2}\right)}$$

b) iii) Since $k^2(k-1)^2$ reaches a maximum at $k = 1/2$, the OLS estimator is highest when $k = 1/2$. For $k > 1/2$, an increase in k (more persistent money growth and inflation), reduces the value of the OLS estimator. As k approaches one — so that money growth, inflation and thus the one-period nominal interest rate all approach random walks — the OLS estimator goes to zero.

Problem 9.7

As described in the text, in equilibrium, output equals $\bar{y}$ and inflation equals $\pi^* + (b/a)(y^* - \bar{y})$.

Substituting these values into the loss function given by equation (9.9) in the text — $L = (1/2)(y - y^*)^2 + (1/2)a(\pi - \pi^*)^2$ — yields the value of the loss function in equilibrium:

$$(1) \quad L^{EQ} = \frac{1}{2}(\bar{y} - y^*)^2 + \frac{1}{2}a \left[\frac{b}{a}(y^* - \bar{y}) \right]^2 = \frac{1}{2}(y^* - \bar{y})^2 + \frac{1}{2} \cdot \frac{b^2}{a}(y^* - \bar{y})^2$$

or simply:

$$(2) \quad L^{EQ} = \frac{1}{2}(y^* - \bar{y})^2 \left[1 + \frac{b^2}{a} \right]$$

Output equals $\bar{y}$ in equilibrium, regardless of the value of a . Thus to see how the equilibrium loss varies with a , use equation (2) to take the derivative of L^{EQ} with respect to a :

$$(3) \quad \frac{\partial L^{EQ}}{\partial a} = \frac{-b^2}{2a^2}(y^* - \bar{y})^2 < 0$$

Equation (3) states that a fall in a increases L^{EQ} . That is, a reduction in the cost of inflation increases the loss to society. It is true that any given deviation in inflation from its optimal level, π^* , has a lower cost to society. However, the problem is that this causes the equilibrium level of inflation itself to be higher.

Intuitively, at a given π^* , the marginal cost of additional inflation is now lower for the policymaker. For a given π^* , it then becomes optimal to set a higher inflation rate for a given π^* . But the public knows this and thus the level of π for which $\pi^e = \pi$ is now higher. It turns out that the fact that π^{EQ} exceeds π^* by more than it used to, outweighs the fact that any given deviation in π^{EQ} from π^* has a lower cost to society.

Problem 9.8

a) Suppose that π differs from $\hat{\pi}$ in some period t_0 . Then $\pi^e = b/a$ for all periods after t_0 . Substituting this expression for expected inflation into the Lucas supply function — $y_t = \bar{y} + b(\pi_t - \pi_t^e)$ — gives us output in each subsequent period:

$$(1) \quad y_t = \bar{y} + b(\pi_t - b/a) \quad \text{for all } t > t_0$$

With expected inflation now constant into the future, the equilibrium in each period is independent of the policymaker's action in the previous period. Thus we can concentrate on a representative period, t — the equilibrium in all periods will be the same. Substituting equation (1) into the policymaker's objective function for period t — $w_t = y_t - (a\pi_t^2/2)$ — yields:

$$(2) \quad w_t = \bar{y} + b(\pi_t - b/a) - a\pi_t^2/2 \quad \text{for all } t > t_0$$

The first-order condition for the choice of inflation is:

$$(3) \quad \partial w_t / \partial \pi_t = b - a\pi_t = 0$$

and thus the policymaker chooses:

$$(4) \quad \pi_t = b/a \quad \text{for all } t > t_0$$

Since $\pi_t = \pi_t^e = b/a$, then from the Lucas supply function,:

$$(5) \quad y_t = \bar{y} \quad \text{for all } t > t_0$$

b) To keep things simple, we can assume that the monetary authority chooses to depart from $\pi = \hat{\pi}$ in period 0. This does not alter the message. Since π has always been equal to $\hat{\pi}$, $\pi_0^e = \hat{\pi}$. Substituting this into the Lucas supply function gives us:

$$(6) \quad y_0 = \bar{y} + b(\pi_0 - b/a)$$

Given the fact that the policymaker is choosing to depart from $\pi = \hat{\pi}$, the particular choice of π_0 does not affect π^e and thus the equilibrium in future periods. Thus only the current period's objective function matters to the policymaker. She will choose π_0 to maximize:

$$(7) w_0 = \bar{y} + b(\pi_0 - \hat{\pi}) - (a\pi_0^2/2)$$

The first-order condition for the choice of π_0 is:

$$(8) \partial w_0 / \partial \pi_0 = b - a\pi_0 = 0$$

and thus the policymaker chooses:

$$(9) \pi_0 = b/a$$

With this choice of inflation, using the Lucas supply function, output in period 0 is given by:

$$(10) y_0 = \bar{y} + b[(b/a) - \hat{\pi}]$$

Substituting equations (9) and (10) into the policymaker's objective function $-w_0 = y_0 - (a\pi_0^2/2)$ yields:

$$(11) w_0 = \bar{y} + (b^2/a) - b\hat{\pi} - (b^2/2a)$$

or simply:

$$(12) w_0 = \bar{y} + (b^2/2a) - b\hat{\pi}$$

As shown in part (a), in all subsequent periods after the policymaker has deviated, $\pi_t = b/a$ and $y_t = \bar{y}$.

Substituting these values of output and inflation into the objective function $-w_t = y_t - (a\pi_t^2/2)$ gives us:

$$(13) w_t = \bar{y} - (b^2/2a) \quad \text{for all } t > 0$$

Thus the policymaker's lifetime objective function if she deviates is given by:

$$(14) W^D = \bar{y} + (b^2/2a) - b\hat{\pi} + \sum_{t=1}^{\infty} \beta^t [\bar{y} - (b^2/2a)]$$

Pulling the $[\bar{y} - (b^2/2a)]$ out of the summation sign and using the fact that, since $\beta < 1$:

$$(15) \sum_{t=1}^{\infty} \beta^t = \beta + \beta^2 + \beta^3 + \dots = \beta(1 + \beta + \beta^2 + \dots) = \beta/(1 - \beta)$$

we can write the lifetime objective function as:

$$(16) W^D = \bar{y} + \frac{b^2}{2a} - b\hat{\pi} + \left(\frac{\beta}{1 - \beta}\right) \left[\bar{y} - \frac{b^2}{2a}\right] = \left(1 + \frac{\beta}{1 - \beta}\right) \bar{y} - b\hat{\pi} + \left(1 - \frac{\beta}{1 - \beta}\right) \frac{b^2}{2a}$$

or simply:

$$(17) W^D = \left(\frac{1}{1 - \beta}\right) \bar{y} - b\hat{\pi} + \left(\frac{1 - 2\beta}{1 - \beta}\right) \frac{b^2}{2a}$$

If the policymaker chooses $\pi = \hat{\pi}$ every period, output will be equal to $\bar{y}$ every period. The policymaker's objective function in each period is therefore given by:

$$(18) w_t = \bar{y} - (a\hat{\pi}^2/2)$$

Thus the policymaker's lifetime objective function if she does not deviate is given by:

$$(19) W^{ND} = \sum_{t=0}^{\infty} \beta^t [\bar{y} - (a\hat{\pi}^2/2)]$$

Pulling the $[\bar{y} - (a\hat{\pi}^2/2)]$ out of the summation sign and using the fact that, since $\beta < 1$:

$$(20) \sum_{t=0}^{\infty} \beta^t = 1 + \beta + \beta^2 + \dots = 1/(1 - \beta)$$

we can write the lifetime objective function as:

$$(21) W^{ND} = \left(\frac{1}{1 - \beta}\right) \left[\bar{y} - \frac{a\hat{\pi}^2}{2}\right]$$

c) One way of solving the problem is to calculate the benefit and cost of deviating as a function of $\hat{\pi}$ and the other parameters. We can then examine the range of $\hat{\pi}$'s over which the cost exceeds the benefit and thus the range of $\hat{\pi}$'s over which the policymaker will choose not to deviate from $\pi = \hat{\pi}$.

The benefit of departing from $\pi = \hat{\pi}$ in some period t_0 is that welfare in period t_0 is $\bar{y} + (b^2/2a) - b\hat{\pi}$ [see equation (12)] rather than $\bar{y} - (a\hat{\pi}^2/2)$ [see equation (18)]. Thus the benefit of deviating, B , is:

$$B = \bar{y} + (b^2/2a) - b\hat{\pi} - \bar{y} + (a\hat{\pi}^2/2)$$

or simply:

$$(22) \quad B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}$$

The cost of deviating is that in all periods subsequent to t_0 , welfare will be $\bar{y} - (b^2/2a)$ [see equation (13)] rather than $\bar{y} - (a\hat{\pi}^2/2)$. Thus the cost of deviating in each future period is $\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + (b^2/2a)$ or simply $(b^2/2a) - (a\hat{\pi}^2/2)$. The total cost of deviating, discounted to time t_0 is:

$$(23) \quad C = \sum_{t=t_0+1}^{\infty} \beta^{t-t_0} [(b^2/2a) - (a\hat{\pi}^2/2)] = [(b^2/2a) - (a\hat{\pi}^2/2)] (\beta + \beta^2 + \beta^3 + \dots)$$

Substituting the result in equation (15) into equation (23) gives us the cost of deviating:

$$(24) \quad C = \left(\frac{\beta}{1-\beta} \right) \left[\frac{b^2}{2a} - \frac{a\hat{\pi}^2}{2} \right]$$

We can plot the benefit and cost from deviating as a function of $\hat{\pi}$. First, we will deal with the benefit from deviating. From equation (22):

$$(25) \quad \partial B / \partial \hat{\pi} = a\hat{\pi} - b \quad \text{and} \quad (26) \quad \partial^2 B / \partial \hat{\pi}^2 = a > 0$$

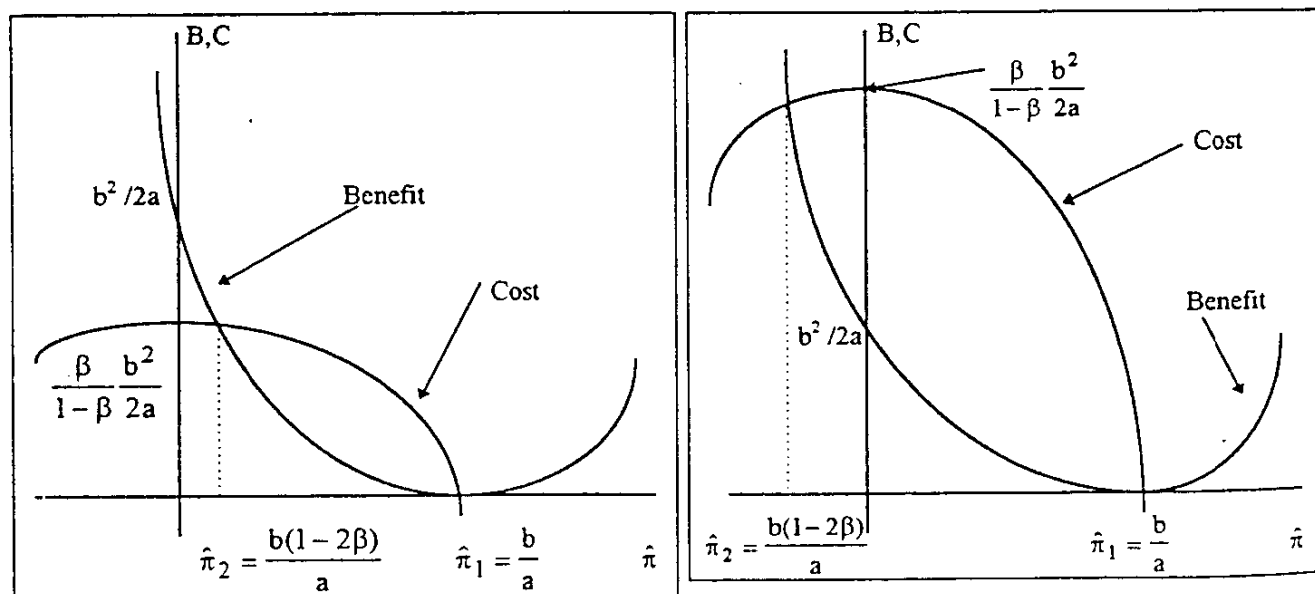
Thus B is a parabola that reaches a minimum at $\hat{\pi} = b/a$. From equation (22), at $\hat{\pi} = 0$, $B = b^2/2a$.

Finally, at its minimum at $\hat{\pi} = b/a$, $B = (b^2/2a) + (b^2/2a) - (b^2/a) = 0$. B — the benefit from deviating as a function of $\hat{\pi}$ — is plotted in both diagrams below.

Now dealing with the cost of deviation, we have from equation (24):

$$(27) \quad \partial C / \partial \hat{\pi} = -[\beta/(1-\beta)]a\hat{\pi} \quad \text{and} \quad (28) \quad \partial^2 C / \partial \hat{\pi}^2 = -[\beta/(1-\beta)]a < 0$$

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. From equation (24), the value of the cost of deviating at $\hat{\pi} = 0$ is given by $C = [\beta/(1-\beta)] \cdot (b^2/2a)$. In addition, at $\hat{\pi} = b/a$, $C = 0$.



The case of $\beta < 1/2$ — so that $\beta/(1 - \beta) < 1$ — is depicted in the left diagram. The case of $\beta > 1/2$ — so that $\beta/(1 - \beta) > 1$ is depicted in the right diagram.

We need to solve for the values of $\hat{\pi}$ where the benefit of deviating equals the cost of deviating. Setting the right-hand sides of equations (22) and (24) equal to each other yields:

$$\frac{b^2}{2a} + \frac{a\hat{\pi}^2}{2} - b\hat{\pi} = \frac{\beta}{1-\beta} \left[\frac{b^2}{2a} - \frac{a\hat{\pi}^2}{2} \right] \Rightarrow \left(1 + \frac{\beta}{1-\beta} \right) \frac{a\hat{\pi}^2}{2} - b\hat{\pi} + \left(1 - \frac{\beta}{1-\beta} \right) \frac{b^2}{2a} = 0$$

or simply:

$$(29) \quad \left(\frac{1}{1-\beta} \right) \frac{a\hat{\pi}^2}{2} - b\hat{\pi} + \left(\frac{1-2\beta}{1-\beta} \right) \frac{b^2}{2a} = 0$$

Multiplying both sides of equation (29) by $(1 - \beta)2a$ gives us an equivalent condition for $B = C$:

$$(30) \quad a^2 \hat{\pi}^2 - 2a(1 - \beta)b\hat{\pi} + (1 - 2\beta)b^2 = 0$$

Using the quadratic formula, we have:

$$(31) \quad \hat{\pi} = \frac{2ab(1 - \beta) \pm \sqrt{4a^2b^2(1 - \beta)^2 - 4a^2b^2(1 - 2\beta)}}{2a^2} = \frac{2ab(1 - \beta) \pm \sqrt{4a^2b^2[1 - 2\beta + \beta^2 - 1 + 2\beta]}}{2a^2}$$

Some further algebra yields:

$$(32) \quad \hat{\pi} = \frac{2ab(1 - \beta) \pm 2ab\beta}{2a^2} = \frac{b(1 - \beta) \pm b\beta}{a}$$

and thus finally:

$$(33) \quad \hat{\pi}_1 = \frac{b(1 - \beta) + b\beta}{a} = \frac{b}{a} \quad \text{and} \quad (34) \quad \hat{\pi}_2 = \frac{b(1 - \beta) - b\beta}{a} = \frac{b(1 - 2\beta)}{a}$$

These two values of $\hat{\pi}$ for which the benefit of deviating just equals the cost of deviating are depicted in the diagrams above. Note that for the case of $\beta > 1/2$ — the diagram on the right — $\hat{\pi}_2$ is negative and is thus not relevant. We can now interpret the diagrams.

For the case of $\beta > 1/2$ — depicted in the diagram on the right — the cost of deviating exceeds the benefit of deviating for any $\hat{\pi}$ such that $0 \leq \hat{\pi} < b/a$. With these values of the parameters, the policymaker will choose not to deviate from $\pi = \hat{\pi}$. Right at $\hat{\pi} = b/a$, the policymaker is indifferent — and in fact at $\hat{\pi} = b/a$, deviating is actually the same as producing $\pi = \hat{\pi}$. Finally, for any value of $\hat{\pi} > b/a$, the benefit of deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

For the case of $\beta < 1/2$ — depicted in the diagram on the left — the cost of deviating exceeds the benefit of deviating for any value of $\hat{\pi}$ such that $[b(1 - 2\beta)]/a < \hat{\pi} < b/a$. With these values of the parameters, the policymaker will choose not to deviate from $\pi = \hat{\pi}$. Right at $\hat{\pi} = b/a$ and $\hat{\pi} = [b(1 - 2\beta)]/a$, the policymaker is indifferent. Finally, for any value of $\hat{\pi} < [b(1 - 2\beta)]/a$ or $\hat{\pi} > b/a$, the benefit of deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

For the policymaker to actually set $\pi = 0$ if $\hat{\pi} = 0$, we would need the cost of deviating to exceed the benefit of deviating, evaluated at $\hat{\pi} = 0$. From our earlier discussion, we know that this will be true as long as $\beta > 1/2$. Thus regardless of the values of a and b , the policymaker will choose to set inflation to 0 if $\hat{\pi} = 0$ as long as the discount rate is greater than $1/2$.

Problem 9.9

a) We can use the same technique as in part (c) of Problem 9.8. We can examine the range of $\hat{\pi}$'s over which the cost of deviating from setting $\pi = \hat{\pi}$ exceeds the benefit of deviating. This gives the range of $\hat{\pi}$'s over which the policymaker chooses $\pi = \hat{\pi}$ each period.

The benefit from deviating, B , is the same as it was in Problem 9.8. Thus we have:

$$(1) B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}$$

The cost of deviating in some period is that in the following period, $\pi^e = b/a$ rather than $\pi^e = \hat{\pi}$. As shown in part (a) of Problem 9.8, when $\pi^e = b/a$, the policymaker chooses $\pi = b/a$. Thus output is equal to $\bar{y}$ in the following period. Note that since the policymaker chooses $\pi = \pi^e$ in the period after deviating, expected inflation reverts to $\pi^e = \hat{\pi}$ in all subsequent periods. Thus there is only a one-period cost to deviating in this setup. Specifically, the cost is that in the period after deviating, the value of the policymaker's objective function is $\bar{y} - (b^2/2a)$ rather than $\bar{y} - (a\hat{\pi}^2/2)$. Discounting that back to the period in which the deviation occurs, yields the cost, C :

$$(2) C = \beta[\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + (b^2/2a)] = \beta[(b^2/2a) - (a\hat{\pi}^2/2)]$$

We can now plot the benefit and cost of deviating as a function of $\hat{\pi}$. The benefit from deviating is the same as in Problem 9.8 and so we can concentrate on the cost.

From equation (2):

$$(3) \partial C / \partial \hat{\pi} = -\beta a \hat{\pi} \quad \text{and} \quad (4) \partial^2 C / \partial \hat{\pi}^2 = -\beta a < 0$$

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. From equation (2), the value of the cost of deviating at $\hat{\pi} = 0$ is given by $C = \beta \cdot (b^2/2a) < (b^2/2a)$ since $\beta < 1$. The next step is to solve for the values of $\hat{\pi}$ where the benefit of deviating equals the cost of deviating. Setting the right-hand sides of equations (1) and (2) equal to each other yields:

$$\frac{b^2}{2a} + \frac{a\hat{\pi}^2}{2} - b\hat{\pi} = \frac{\beta b^2}{2a} - \frac{\beta a\hat{\pi}^2}{2} \Rightarrow \frac{(1+\beta)a}{2} \hat{\pi}^2 - b\hat{\pi} + \frac{(1-\beta)b^2}{2a} = 0 \quad (5)$$

Multiplying both sides of equation (5) by $2a$ gives us an equivalent condition for $B = C$:

$$(6) (1+\beta)a^2 \hat{\pi}^2 - 2ab\hat{\pi} + (1-\beta)b^2 = 0$$

Using the quadratic formula, we have:

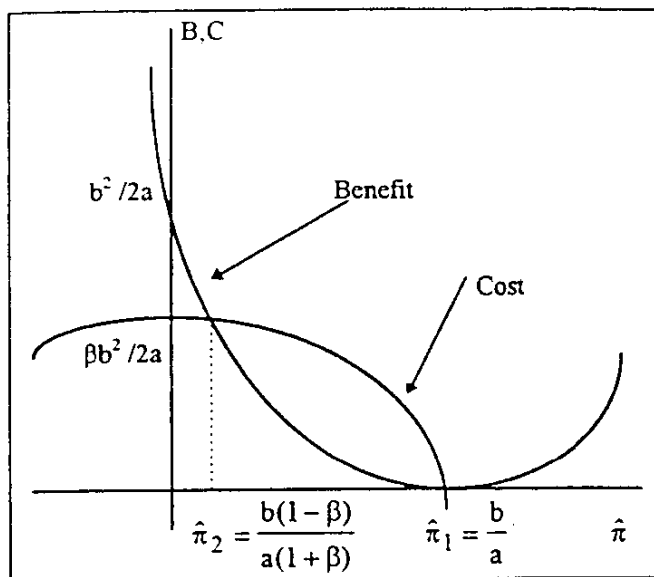
$$(7) \hat{\pi} = \frac{2ab \pm \sqrt{4a^2b^2 - 4a^2b^2(1+\beta)(1-\beta)}}{2a^2(1+\beta)} = \frac{2ab \pm \sqrt{4a^2b^2[1 - (1+\beta)(1-\beta)]}}{2a^2(1+\beta)}$$

Some further algebra yields:

$$(8) \hat{\pi} = \frac{2ab \pm 2ab\beta}{2a^2(1+\beta)} = \frac{b(1 \pm \beta)}{a(1+\beta)}$$

and thus finally:

$$(9) \hat{\pi}_1 = \frac{b(1+\beta)}{a(1+\beta)} = \frac{b}{a} \quad \text{and} \quad (10) \hat{\pi}_2 = \frac{b(1-\beta)}{a(1+\beta)}$$



From the diagram at left, we can see that the cost of deviating from $\pi = \hat{\pi}$ exceeds the benefit from deviating for any $\hat{\pi}$ such that $b(1 - \beta)/a(1 + \beta) < \hat{\pi} < b/a$. With these values of the parameters, the policymaker will choose not to deviate. For any value of $\hat{\pi}$ greater than b/a or less than $b(1 - \beta)/a(1 + \beta)$, the benefit from deviating exceeds the cost of deviating and hence the policymaker will in fact deviate from $\pi = \hat{\pi}$.

- b) Again, we will employ the same technique. The benefit from deviating remains the same; it is given by:
- (1) $B = (b^2/2a) + (a\hat{\pi}^2/2) - b\hat{\pi}$

We need to determine the cost of deviating for the policymaker. Suppose the policymaker deviates in some period t . Then in period $t + 1$, $\pi_{t+1}^e = \pi_0 > b/a$. We can also write this as $\pi_{t+1}^e = b/a + x$, $x > 0$. The variable x will capture the extent of the punishment for deviating. When the policymaker takes expected inflation as given, she chooses to set inflation to b/a . Thus, using the Lucas supply function, output in period $t + 1$ – the period after deviating – is:

$$(11) \quad y_{t+1} = \bar{y} + b[(b/a) - (b/a) - x] = \bar{y} - bx$$

Thus output is below the natural rate the period after deviating. The value of the policymaker's objective function in period $t + 1$ is:

$$(12) \quad w_{t+1} = y_{t+1} - (a\pi^2/2) = \bar{y} - bx - (b^2/2a)$$

Thus the cost of deviating in period $t + 1$ is that welfare is given by (12) rather than being equal to $\bar{y} - (a\hat{\pi}^2/2)$. Discounting this back to period t , we have the cost in period $t + 1$:

$$(13) \quad C_{t+1} = \beta[\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + bx + (b^2/2a)]$$

or simply:

$$(14) \quad C_{t+1} = \beta[bx - (a\hat{\pi}^2/2) + (b^2/2a)]$$

Now consider the situation in period $t + 2$, two periods after a deviation. Expected inflation equals b/a . Taking expected inflation as given, the policymaker chooses inflation equal to b/a . Thus output is at the natural rate. The value of the policymaker's objective function in $t + 2$ is:

$$(15) \quad w_{t+2} = y_{t+2} - (a\pi^2/2) = \bar{y} - (b^2/2a)$$

Thus the cost of deviating in period $t + 2$ is that welfare is equal to $\bar{y} - (b^2/2a)$ rather than $\bar{y} - (a\hat{\pi}^2/2)$.

Discounting this back to period t , we have the cost in period $t + 2$:

$$(16) \quad C_{t+2} = \beta^2[\bar{y} - (a\hat{\pi}^2/2) - \bar{y} + (b^2/2a)] = \beta^2[(b^2/2a) - (a\hat{\pi}^2/2)]$$

In period $t + 3$, since actual inflation last period equaled expected inflation last period, expected inflation reverts to $\hat{\pi}$ and there is no further cost to the deviation in period t .

Thus the total cost of the deviation is:

$$(17) C = \beta[bx - (a\hat{\pi}^2/2) + (b^2/2a)] + \beta^2[(b^2/2a) - (a\hat{\pi}^2/2)]$$

or simply:

$$(18) C = \beta bx + \beta(1 + \beta)[(b^2/2a) - (a\hat{\pi}^2/2)]$$

From equation (18):

$$(19) \partial C / \partial \hat{\pi} = \beta(1 + \beta)[-2a\hat{\pi}/2] = -\beta(1 + \beta)a\hat{\pi} \quad \text{and} \quad (20) \partial^2 C / \partial \hat{\pi}^2 = -\beta(1 + \beta)a < 0$$

Thus C is an inverted parabola that reaches a maximum at $\hat{\pi} = 0$. The value of the cost of deviating at $\hat{\pi} = 0$ is given by $\beta bx + \beta(1 + \beta)(b^2/2a)$. From earlier analysis, we know that the benefit of deviating at $\hat{\pi} = 0$ is $b^2/2a$. Thus if the value of x — the excess punishment — is high enough, the cost of deviating at $\hat{\pi} = 0$ will exceed the benefit and there can be an equilibrium with zero inflation. Specifically, we need the following condition to hold:

$$(20) \beta bx + \beta(1 + \beta)(b^2/2a) > b^2/2a$$

or:

$$(21) \beta bx > (b^2/2a)[1 - \beta(1 + \beta)]$$

or simply:

$$(22) x > (b^2/2a)[1 - \beta(1 + \beta)]/\beta b$$

We can determine the value of $\hat{\pi}$ at which $C = 0$. From equation (18), $C = 0$ when:

$$(23) \beta(1 + \beta)[(a\hat{\pi}^2/2) - (b^2/2a)] = \beta bx$$

which implies:

$$(24) (a\hat{\pi}^2/2) - (b^2/2a) = bx/(1 + \beta)$$

and thus:

$$(25) \hat{\pi}^2 = [2bx/a(1 + \beta)] + (b^2/a^2)$$

Therefore $C = 0$ when:

$$(26) \hat{\pi} = \sqrt{\frac{2bx}{a(1 + \beta)} + \frac{b^2}{a^2}} > \frac{b}{a}$$

Thus the cost of deviating is equal to zero at a value of $\hat{\pi}$ greater than the one for which the benefit from deviating is equal to zero (which is $\hat{\pi} = b/a$). The values of $\hat{\pi}$ for which it is an equilibrium for the policymaker not to deviate are those — just as in part (a) — where the cost of deviating exceeds the benefit. The basic idea here is that higher values of x lead to a wider range of $\hat{\pi}$'s for which the cost exceeds the benefit and thus the wider the range of $\hat{\pi}$'s for which the policymaker does not deviate.

c) As we have shown previously, if the policymaker takes expected inflation as given, she chooses inflation equal to b/a . Thus if $\pi^e = b/a$, the policymaker chooses $\pi = b/a$, so that the public's expectation is fulfilled and output is at the natural rate. There is no incentive for the policymaker to choose a different inflation rate and there is no incentive for the public to change its expectation of inflation and thus $\pi = \pi^e = b/a$ will be an equilibrium for any $a > 0$, $b > 0$.

Problem 9.10

Consider the situation in the last period, denoted T . The policymaker's choice of π has no effect on next period's expected inflation — there is no next period. Thus the policymaker's problem in the final period is to take expected inflation as given and choose π in order to maximize the period T objective function. From previous analysis in Problem 9.8, we know that the policymaker's choice of inflation in this type of situation is $\pi_T = b/a$. Since the public knows how the policymaker behaves, expected inflation also equals b/a and thus output equals $\bar{y}$.

Now consider the situation in period $T - 1$. The important point is that the policymaker knows her choice of π_{T-1} will have no bearing on what happens the next and final period. Regardless of the level of π she chooses in period $T - 1$, expected inflation next period will be b/a , as described above. Since the policymaker's problem has no impact on the future, she chooses π -- taking π^e as given -- in order to maximize the period $T - 1$ objective function. Again, the optimal choice is $\pi_{T-1} = b/a$. The public knows this and so $\pi_{T-1}^e = b/a$ and thus output in period $T - 1$ equals $\bar{y}$.

Working backward, the same thing happens each period. The policymaker knows that expected inflation the following period will be b/a regardless of what she does this period. Thus she acts to maximize the one-period objective function and chooses $\pi = b/a$, which results in output equal to the natural rate. Therefore the unique equilibrium for all periods is $\pi_t^e = \pi_t = b/a$ and $y_t = \bar{y}$.

Problem 9.11

a) The policymaker wants to choose $\hat{\pi}_1$ and $\hat{\pi}_2$ in order to maximize the objective function given by:

$$(1) W = E[b(\pi_1 - \pi_1^e) + c\pi_1 - (a\pi_1^2/2) + b(\pi_2 - \pi_2^e) + c\pi_2 - (a\pi_2^2/2)]$$

Substituting $\pi_1 = \hat{\pi}_1 + \varepsilon_1$ and $\pi_2^e = \alpha + \beta\pi_1$ into the objective function given in equation (1) yields:

$$(2) W = E\{b(\hat{\pi}_1 + \varepsilon_1 - \pi_1^e) + c(\hat{\pi}_1 + \varepsilon_1) - [a(\hat{\pi}_1 + \varepsilon_1)^2/2] + b[\hat{\pi}_2 + \varepsilon_2 - \alpha - \beta(\hat{\pi}_1 + \varepsilon_1)] + c(\hat{\pi}_2 + \varepsilon_2) - [a(\hat{\pi}_2 + \varepsilon_2)^2/2]\}$$

The first-order condition for the policymaker's choice of $\hat{\pi}_2$ is:

$$(3) \partial W / \partial \hat{\pi}_2 = E(b + c - a\hat{\pi}_2) = 0$$

Solving for $\hat{\pi}_2$, noting that c is not uncertain from the policymaker's perspective, we have:

$$(4) \hat{\pi}_2 = (b + c)/a$$

Substituting equation (4) into the expected value of the policymaker's second-period objective function yields:

$$(5) E(w_2) = E\left[b\left(\frac{b+c}{a} + \varepsilon_2 - \pi_2^e\right) + c\left(\frac{b+c}{a} + \varepsilon_2\right) - \frac{a\left[(b+c)/a + \varepsilon_2\right]^2}{2}\right]$$

Since this expectation is being taken with respect to the policymaker's information set, c is not random.

Thus equation (5) can be rewritten as:

$$(6) E(w_2) = \frac{b(b+c)}{a} - b\pi_2^e + \frac{c(b+c)}{a} - \frac{a}{2}\left[\frac{(b+c)^2}{a^2} + \frac{2(b+c)E(\varepsilon_2)}{a} + E(\varepsilon_2^2)\right]$$

Note that $E(\varepsilon_2) = 0$ and $E(\varepsilon_2^2) = \sigma_\varepsilon^2$. Thus equation (6) simplifies to:

$$(7) E(w_2) = \frac{(b+c)^2}{a} - b\pi_2^e - \frac{(b+c)^2}{2a} - \frac{a\sigma_\varepsilon^2}{2}$$

and thus finally:

$$(8) E(w_2) = \frac{(b+c)^2}{2a} - \frac{a\sigma_\varepsilon^2}{2} - b\pi_2^e$$

b) Using equation (2), the first-order condition for the policymaker's choice of $\hat{\pi}_1$ is:

$$(9) \partial W / \partial \hat{\pi}_1 = E(b + c - a\hat{\pi}_1 - b\beta) = 0$$

Since nothing on the right-hand side of equation (9) is uncertain for the policymaker, solving for $\hat{\pi}_1$ gives us:

$$(10) \hat{\pi}_1 = [b(1 - \beta) + c]/a$$

c) π_1 and π_2 are linear functions of c and ε , which are normal random variables. Thus π_1 and π_2 are also normal random variables. Therefore we can use the following formula for the conditional expectation of a normal:

$$(11) \quad E[\pi_2 | \pi_1] = E(\pi_2) + \frac{\text{cov}(\pi_2, \pi_1)}{\text{var}(\pi_1)} \cdot [\pi_1 - E(\pi_1)]$$

Equation (11) is intuitive. Suppose that π_1 and π_2 have a positive covariance — which means that π_2 tends to be above its mean when π_1 is above its mean. Then if we observe a realization of π_1 greater than its expected value, the second term on the right-hand side of equation (11) is positive. This means that given this realization of π_1 , we should expect the realization of π_2 to be greater than its unconditional mean of $E(\pi_2)$.

We need to solve for $E(\pi_1)$, $E(\pi_2)$, $\text{cov}(\pi_2, \pi_1)$ and $\text{var}(\pi_1)$. We know that $\pi_1 = \hat{\pi}_1 + \varepsilon_1$ and thus:

$$(12) \quad E(\pi_1) = E(\hat{\pi}_1) + E(\varepsilon_1) = [b(1 - \beta) + \bar{c}]/a$$

The last step uses the fact that the public knows that the policymaker will choose $\hat{\pi}_1$ according to equation (10). Thus with rational expectations, the expected value of $\hat{\pi}_1$ must equal the expected value of the right-hand side of equation (10). In addition, the last step uses the fact that $E(\varepsilon_1) = 0$.

Since $\pi_2 = \hat{\pi}_2 + \varepsilon_2$, we have:

$$(13) \quad E(\pi_2) = E(\hat{\pi}_2) + E(\varepsilon_2) = (b + \bar{c})/a$$

Again, the last step uses the fact that the public knows that the policymaker will choose $\hat{\pi}_2$ according to equation (4). Thus with rational expectations, the expected value of $\hat{\pi}_2$ must equal the expected value of the right-hand side of equation (4). In addition, the last step uses the fact that $E(\varepsilon_2) = 0$.

Now we need to find the variance of inflation in period 1:

$$(14) \quad \text{var}(\pi_1) = \text{var}(\hat{\pi}_1 + \varepsilon_1) = \text{var}([b(1 - \beta) + c]/a + \varepsilon_1)$$

Since c and ε_1 are independent, we have:

$$(15) \quad \text{var}(\pi_1) = (1/a^2) \sigma_c^2 + \sigma_\varepsilon^2$$

Finally, the covariance between π_1 and π_2 is given by:

$$(16) \quad \text{cov}(\pi_1, \pi_2) = \text{cov}([b(1 - \beta) + c]/a + \varepsilon_1, (b + c)/a + \varepsilon_2)$$

Since ε_1 , ε_2 and c are independent, this covariance is equal to:

$$(17) \quad \text{cov}(\pi_1, \pi_2) = \text{cov}(c/a, c/a) = \text{var}(c/a) = (1/a^2) \sigma_c^2$$

Substituting equations (12), (13), (15) and (17) into equation (11) yields:

$$(18) \quad E[\pi_2 | \pi_1] = \frac{(b + \bar{c})}{a} + \frac{(1/a^2) \cdot \sigma_c^2}{(1/a^2) \cdot \sigma_c^2 + \sigma_\varepsilon^2} \cdot \left[\pi_1 - \frac{b(1 - \beta) + \bar{c}}{a} \right]$$

and thus β is given by:

$$(19) \quad \beta = \frac{(1/a^2) \cdot \sigma_c^2}{(1/a^2) \cdot \sigma_c^2 + \sigma_\varepsilon^2}$$

The intuition behind equation (18) is as follows. The public wants to form its expectation of inflation in period 2, given its observation of inflation in period 1. In order to do so, the public would like to know for sure what the policymaker's taste for inflation, c , is. The problem is that actual inflation in period 1 does depend on what the true c is, but it also depends upon the random, unobservable ε shock. Now, if the public sees a π_1 greater than its expected value of $[b(1 - \beta) + \bar{c}]/a$, it knows this could be due to a policymaker with a higher than average c . If this is the case, the public should revise upward its estimate of π_2 from $(b + \bar{c})/a$, its unconditional mean. However, this π_1 greater than its expected value could also be due to a positive realization of ε_1 . If this is the case, this should have no bearing on the public's estimate

of π_2 . Equation (18) says that if the variance of the policymaker's taste for inflation σ_c^2 , is very large relative to the variance of the random shocks, σ_ϵ^2 , β will be close to 1. The public will attribute most of the above average realization of π_1 to a policymaker with a higher than average c and raise the expectation of π_2 accordingly.

d) The policymaker knows that her choice of $\hat{\pi}_1$ will affect the public's expectation of inflation in the second period, π_2^e . When π_1 turns out to be high, the public attributes some of this to a policymaker with a high c and accordingly raise π_2^e . From equation (8), we can see that a higher value of π_2^e reduces the expected value of the policymaker's second-period objective function. Thus the policymaker chooses a lower $\hat{\pi}_1$ to try and establish a "good reputation" as someone with a low c in order to keep π_2^e down. In the second period, however, there is no future period. Thus there is no need to worry about the effects that this period's inflation will have on future expected inflation.

Problem 9.12

a) The policymaker wants to choose inflation in order to maximize her objective function — $W = c\gamma y - (a\pi^2/2)$ — subject to output being given by the Lucas supply function — $y = \bar{y} + b(\pi - \pi^e)$.

Thus the policymaker's problem is:

$$\max_{\pi} W = c\gamma[\bar{y} + b(\pi - \pi^e)] - (a\pi^2/2)$$

The first-order condition is:

$$(1) \partial W/\partial \pi = bc\gamma - a\pi = 0$$

Thus the policymaker's choice of π is:

$$(2) \pi = bc\gamma/a$$

b) The public knows the policymaker will set inflation according to equation (2). Thus with rational expectations, expected inflation must equal the expectation of the right-hand side of equation (2):

$$(3) \pi^e = E[bc\gamma/a] = bcE(\gamma)/a = bc\bar{\gamma}/a$$

c) The true social welfare function is given by $W^{SOC} = \gamma y - (a\pi^2/2)$. Taking the expectation of both sides of this expression with respect to the public's information set, so that γ is random, we have:

$$(4) E(W^{SOC}) = E\{\gamma[\bar{y} + b(\pi - \pi^e)] - (a\pi^2/2)\}$$

where we have substituted for $y = \bar{y} + b(\pi - \pi^e)$. Now substitute the policymaker's choice of π , equation (2), and the public's expectation of inflation, equation (3), into equation (4):

$$(5) E(W^{SOC}) = E\left\{\gamma\left[\bar{y} + b\left(\frac{bc\gamma}{a} - \frac{bc\bar{\gamma}}{a}\right)\right] - \frac{ab^2c^2\gamma^2}{2a^2}\right\}$$

Simplifying yields:

$$(6) E(W^{SOC}) = \bar{\gamma}E(\gamma) + \frac{b^2cE(\gamma^2)}{a} - \frac{b^2c\bar{\gamma}E(\gamma)}{a} - \frac{b^2c^2E(\gamma^2)}{2a}$$

Since $E(\gamma) = \bar{\gamma}$, equation (6) becomes:

$$(7) E(W^{SOC}) = \bar{\gamma} \cdot \bar{\gamma} + \frac{b^2c}{a}[E(\gamma^2) - \bar{\gamma}^2] - \frac{b^2c^2E(\gamma^2)}{2a}$$

Now use the facts that for a random variable X :

$$(8) \text{var}(X) = E(X^2) - [E(X)]^2 \quad \text{and} \quad (9) E(X^2) = \text{var}(X) + [E(X)]^2$$

Here, this means that we can write:

$$(10) \sigma_\gamma^2 = E(\gamma^2) - \bar{\gamma}^2 \quad \text{and} \quad (11) E(\gamma^2) = \sigma_\gamma^2 + \bar{\gamma}^2$$

Substituting equations (10) and (11) into equation (7) gives us the expected value of the true social welfare function:

$$(12) \quad E(W^{SOC}) = \bar{y} \cdot \bar{\gamma} + \frac{b^2 c}{a} \cdot \sigma_\gamma^2 - \frac{b^2 c^2}{2a} \cdot (\sigma_\gamma^2 + \bar{\gamma}^2)$$

d) To find the first-order condition for the maximization, use equation (12) to set the derivative of the expected value of the social welfare function with respect to c equal to zero:

$$(13) \quad \frac{\partial E(W^{SOC})}{\partial c} = \frac{b^2}{a} \cdot \sigma_\gamma^2 - \frac{b^2 c}{a} \cdot (\sigma_\gamma^2 + \bar{\gamma}^2) = 0$$

Solving for c yields:

$$(14) \quad c = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \bar{\gamma}^2}$$

There is a tradeoff here. From equation (2), we can see that choosing a more "conservative" policymaker — one with a low c — produces a better performance in terms of average inflation. However, such a policymaker would not respond well to the shocks. Thus there is some optimal level of "conservatism" that balances these two forces.

The value of c that maximizes the expected value of true social welfare is decreasing in the mean of γ . Since we know that π^e will equal π on average (since γ will equal $\bar{\gamma}$ on average), output will equal full employment output on average, regardless of the values of c or $\bar{\gamma}$. From equation (2), we can see that if γ is higher on average, inflation will also be higher on average, for a given c . Thus it will be welfare improving to offset this and keep inflation lower on average by having a policymaker with a lower c — that is, having a more "conservative" policymaker.

However, the value of c that maximizes expected social welfare is increasing in the variance of the γ shock. The more variable is the shock, the less "conservative" the central banker should be. Since the policymaker can act after γ is realized, she can choose to offset any deviation in γ from its expected value, which will raise welfare. The policymaker will do this only to the extent that she cares about the shock's effect. Thus the more that γ varies, the better it is to have a policymaker who cares about the shock's effect and will act to offset it.

Problem 9.13

a) Social welfare is higher when the policymaker turns out to be a Type-1 — the type that shares the public's preferences concerning output and inflation. The choice of setting $\pi = 0$ in both periods — as the Type-2 policymaker does — is a choice available to the Type-1 policymaker. She chooses not to do this — in order to maximize social welfare, she decides to choose another pair of inflation rates. Since she is attempting to maximize social welfare, welfare must be higher under the choices made by the Type-1 policymaker. For example, as explained in the text, if $\beta < 1/2$, it is optimal for the Type-1 policymaker to choose $\pi_1 = b/a$ and $\pi_2 = b/a$. That must be because it achieves higher welfare than choosing $\pi_1 = 0$, $\pi_2 = 0$.

b) Expected inflation, π^e , is determined by the public's beliefs. So both the "a'" policymaker and the "a" policymaker face the same π^e , since in either case, the public believes it is facing an a' policymaker. Thus both policymakers have the same choice set. The "a" policymaker makes her choice to maximize true social welfare, whereas the "a'" policymaker makes her choice to maximize something else. Thus social welfare must be higher with the a policymaker.

Problem 9.14

a) When the policymaker fixes i , the LM curve is irrelevant. Equilibrium output is determined by the IS curve and the fixed nominal interest rate, $\bar{i}$. Substituting $\bar{i}$ into the IS curve yields:

$$(1) \quad y = c - a\bar{i} + \varepsilon_{IS}$$

The variance of y is simply:

$$(2) \quad \text{var}(y) = \text{var}(\varepsilon_{IS}) = \sigma_{IS}^2$$

b) When the policymaker fixes m , the equilibrium level of output is determined by the intersection of the IS and LM curves. Rearranging the IS curve to solve for i gives us:

$$(3) \quad i = (c + \varepsilon_{IS} - y)/a$$

Substituting equation (3) and the assumption that $m = \bar{m}$ into the LM curve -- $m - p = hy - ki + \varepsilon_{LM}$ -- gives us:

$$\bar{m} - p = hy - [k(c + \varepsilon_{IS} - y)/a] + \varepsilon_{LM} = [h + (k/a)]y - (kc/a) - (k/a)\varepsilon_{IS} + \varepsilon_{LM}$$

Solving for y yields:

$$(4) \quad y = \frac{\bar{m} - p + (kc/a) + (k/a)\varepsilon_{IS} + \varepsilon_{LM}}{h + (k/a)} = \frac{a(\bar{m} - p) + kc + k\varepsilon_{IS} + a\varepsilon_{LM}}{ah + k}$$

The variance of y is:

$$(5) \quad \text{var}(y) = \left(\frac{k}{ah + k}\right)^2 \sigma_{IS}^2 + \left(\frac{a}{ah + k}\right)^2 \sigma_{LM}^2$$

c) If $\sigma_{IS}^2 = 0$ -- if there are only LM shocks -- then from equations (2) and (5):

$$(6) \quad \text{var}(y)|_{i=\bar{i}} = 0 \quad \text{and} \quad (7) \quad \text{var}(y)|_{m=\bar{m}} = \left(\frac{a}{ah + k}\right)^2 \sigma_{LM}^2 > 0$$

Thus interest-rate targeting leads to a lower variance of output than money-stock targeting. In fact, output is constant under interest targeting.

d) If $\sigma_{LM}^2 = 0$ -- if there are only IS shocks -- then from equations (2) and (5):

$$(8) \quad \text{var}(y)|_{i=\bar{i}} = \sigma_{IS}^2 \quad \text{and} \quad (9) \quad \text{var}(y)|_{m=\bar{m}} = \left(\frac{k}{ah + k}\right)^2 \sigma_{IS}^2 < \sigma_{IS}^2$$

Thus money-stock targeting leads to a lower variance of output than interest-rate targeting.

e) Consider the situation in part c where there are only LM shocks. If the policymaker targets the nominal money stock, the LM shocks cause the LM curve to shift around and equilibrium output in the economy is determined by the intersection of that shifting LM curve with the stable IS curve. If the policymaker targets the nominal interest rate, it ensures that i remains constant in the face of any LM shock. Since i is not allowed to change, planned expenditure does not change and thus the level of output that equates planned and actual expenditure does not change in the face of an LM shock.

Consider the situation in part d where there are only IS shocks. If the policymaker targets the nominal interest rate, equilibrium output changes by the full extent of the shift in the IS curve caused by a shock to the IS curve. Now consider the case where the policymaker targets the nominal money stock. A positive IS shock shifts the IS curve to the right. With m fixed, as Y rises to equate planned and actual expenditure, i rises as well in order for the money market to remain in equilibrium. This rise in i reduces planned expenditure and thus mitigates some of the positive shock. Therefore Y does not end up rising as much.

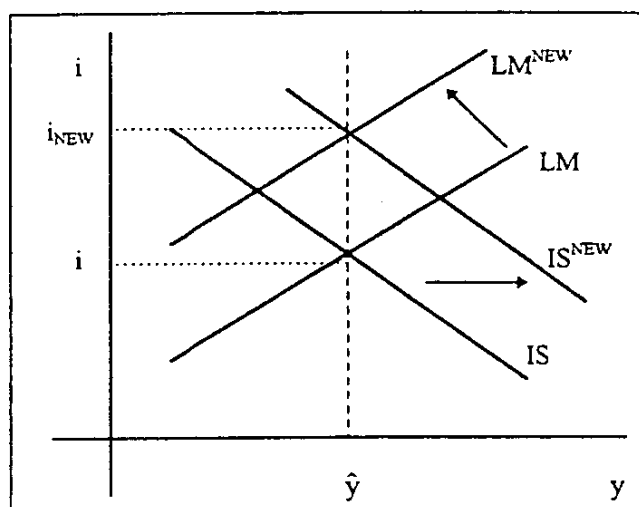
The same idea is true in the opposite direction. A negative IS shock shifts IS to the left. If the policymaker targets m , i will fall along with Y in order to keep the money market in equilibrium. This fall in i raises planned expenditure and helps to offset the original negative shock to planned expenditure. Thus Y does not fall as much as if the policymaker had kept i constant.

f) If there are only IS shocks, it is possible to keep y constant at some target level $\hat{y}$. By rearranging the LM curve with y set to $\hat{y}$, the nominal money supply must be such that:

$$(10) \quad m \Rightarrow p + h\hat{y} - ki$$

The policymaker knows the fixed p , has picked $\hat{y}$ herself and can observe i . Thus when i changes -- and since there are only IS shocks, we know this must be coming from a shift of the IS curve -- the policymaker must change m accordingly. As i rises, for example, the policymaker must reduce m .

In the diagram at right, as i rises due to the rightward shift of the IS curve, the policymaker reduces m which shifts up the LM curve and increases i more. The policymaker can stop reducing m when m and i are such that equation (10) is satisfied. At that point, LM^{NEW} will intersect IS^{NEW} right at the target level of $\hat{y}$.



Problem 9.15

a) Using the fact that for a random variable X , $\text{var}(X) = E(X^2) - [E(X)]^2$ or $E(X^2) = \text{var}(X) + [E(X)]^2$, we have:

$$(1) \quad E[(y - y^*)^2] = \text{var}(y - y^*) + [E(y - y^*)]^2$$

Substituting the expression for output -- $y = x + (k + \varepsilon_k)z + u$ -- into $\text{var}(y - y^*)$ and simplifying yields:

$$(2) \quad \text{var}(y - y^*) = \text{var}(x + kz + \varepsilon_k z + u - y^*) = z^2 \sigma_k^2 + \sigma_u^2$$

Substituting for output in $[E(y - y^*)]^2$ and simplifying yields:

$$(3) \quad [E(y - y^*)]^2 = [E(x + kz + \varepsilon_k z + u - y^*)]^2 = (x + kz - y^*)^2$$

where the last step uses the fact that ε_k and u both have mean 0. Substituting equations (2) and (3) into equation (1) gives us:

$$(4) \quad E[(y - y^*)^2] = z^2 \sigma_k^2 + \sigma_u^2 + (x + kz - y^*)^2$$

b) The policymaker wants to choose z in order to minimize $E[(y - y^*)^2]$. Using equation (4), the first-order condition is:

$$(5) \quad \partial[E(y - y^*)^2] / \partial z = 2z\sigma_k^2 + 2k(x + kz - y^*) = 0$$

Collecting the terms in z yields:

$$z(\sigma_k^2 + k^2) = (y^* - x)k$$

and thus the optimal choice of z is:

$$(6) \quad z = \frac{(y^* - x)k}{\sigma_k^2 + k^2}$$

c) To see how policy should respond to shocks (i.e. changes in x), use equation (6) to take the derivative of z with respect to x :

$$(7) \frac{\partial z}{\partial x} = -\frac{k}{\sigma_k^2 + k^2} < 0$$

The fact that the derivative in (7) is negative simply means that higher values of x should be offset with lower values of z in order to keep output from varying as much.

Note that $\partial z/\partial x$ does not depend upon σ_a^2 , which represents uncertainty about the state of the economy. Thus in this model, the optimal degree of "fine-tuning" does not depend upon the amount of uncertainty about the state of the economy.

d) In contrast, $\partial z/\partial x$ does depend upon σ_k^2 , which represents uncertainty about the effects of the policy instrument. In fact:

$$(8) \frac{\partial[\partial z/\partial x]}{\partial \sigma_k^2} = \frac{k}{(\sigma_k^2 + k^2)^2} > 0$$

Since $\partial z/\partial x$ is negative to begin with, a rise in σ_k^2 makes it less negative. That is, higher values of σ_k^2 — more uncertainty about the effects of the policy instrument — reduces the amount that policy should respond to shocks or in other words, reduces the amount of "fine-tuning" that should be done.

Problem 9.16

We can focus on a situation where g_M , π , i and r are constant and where $\pi^e = \pi$. Although not technically correct — since Y and thus M/P are growing — such a situation will be referred to as a steady state in what follows. Under these assumptions, it is therefore reasonable to assume that output, and the real interest rate are unaffected by the rate of money growth and that actual and expected inflation are equal. Taking the exponential function of both sides of the money demand function — $\ln(M(t)/P(t)) = a - bi + \ln Y(t)$ — yields:

$$(1) M(t)/P(t) = e^{a-bi} Y(t)$$

The nominal interest rate is $i = r + \pi^e$. In steady state, π^e and r are constant and thus so is the nominal interest rate. Thus in steady state, the quantity of real balances must grow at the same rate as $Y(t)$. In other words, $\dot{M}(t)/M(t) - \dot{P}(t)/P(t) = g_Y$. Solving for inflation yields:

$$(2) \pi = g_M - g_Y$$

where g_M is the growth rate of the nominal money stock. This means that the nominal interest rate in steady state is given by:

$$(3) i = \bar{r} + \pi = \bar{r} + g_M - g_Y$$

where we have used the fact that actual and expected inflation are equal. Substituting equation (3) into equation (1) gives steady-state real balances:

$$(4) M(t)/P(t) = e^a e^{-b(\bar{r}+g_M-g_Y)} Y(t)$$

Seignorage is given by:

$$(5) S(t) = \frac{\dot{M}(t)}{P(t)} = \frac{\dot{M}(t)}{M(t)} \cdot \frac{M(t)}{P(t)} = g_M \cdot \frac{M(t)}{P(t)}$$

Substituting equation (4) into equation (5) gives steady-state seignorage:

$$(6) S(t) = g_M e^a e^{-b(\bar{r}+g_M-g_Y)} Y(t) = C g_M e^{-bg_M} Y(t)$$

where $C \equiv e^a e^{-b(\bar{r}-g_Y)}$. We can now find the choice of nominal money growth, g_M , that maximizes steady-state seignorage. Again, we are assuming that output is unaffected by money growth. The first-order condition is:

$$(7) \partial S(t)/\partial g_M = Ce^{-bg_M} Y(t) - bCg_M e^{-bg_M} Y(t) = 0$$

which simplifies to:

$$(8) 1 - bg_M = 0$$

Thus seignorage is maximized when money growth is given by:

$$(9) g_M = 1/b$$

From equation (2), we know that $\pi = g_M - g_Y$ and thus the rate of inflation that maximizes seignorage is:

$$(10) \pi = (1/b) - g_Y$$

Equation (10) implies that the higher is the growth rate of real output, the lower is the rate of inflation that maximizes steady-state seignorage.

Problem 9.17

a) Desired real money holdings are given by:

$$(1) m(t) = Ce^{-b\pi^e(t)}$$

The assumption is that expected inflation adjusts gradually toward actual inflation. Specifically, our assumption is:

$$(2) \dot{\pi}^e(t) = \beta[\pi(t) - \pi^e(t)]$$

As usual, seignorage is given by $\dot{M}(t)/P(t)$ or equivalently $[\dot{M}(t)/M(t)][M(t)/P(t)]$. Assuming that the nominal money supply is growing at rate $g_M(t)$, we can write seignorage as:

$$(3) S(t) = g_M(t)m(t)$$

To see the dynamics of inflation and money holdings formally, note that the growth rate of real money, $\dot{m}(t)/m(t)$, equals the growth rate of nominal money, $g_M(t)$, minus the rate of inflation, $\pi(t)$. Rewriting this as an equation for inflation gives us:

$$(4) \pi(t) = g_M(t) - [\dot{m}(t)/m(t)]$$

Define G as the amount of real purchases that the government needs to finance with seignorage. Thus from equation (3), we have:

$$(5) g_M(t) = G/m(t)$$

Taking the time derivative of both sides of equation (1) yields:

$$(6) \dot{m}(t) = -b\dot{\pi}^e(t)Ce^{-b\pi^e(t)}$$

Dividing both sides of equation (6) by $m(t)$ gives us:

$$(7) \dot{m}(t)/m(t) = -b\dot{\pi}^e(t)$$

Substituting equations (5) and (7) into equation (4) yields:

$$(8) \pi(t) = \frac{G}{m(t)} + b\dot{\pi}^e(t)$$

Substituting equation (8) into equation (2) gives us:

$$(9) \dot{\pi}^e(t) = \beta \left[\frac{G}{m(t)} + b\dot{\pi}^e(t) - \pi^e(t) \right]$$

Collecting the terms in $\dot{\pi}^e(t)$ yields:

$$\dot{\pi}^e(t)[1 - \beta b] = \beta \left[\frac{G}{m(t)} - \pi^e(t) \right]$$

and thus:

$$(10) \dot{\pi}^e(t) = \frac{\beta}{1 - \beta b} \left[\frac{G - \pi^e(t)m(t)}{m(t)} \right]$$

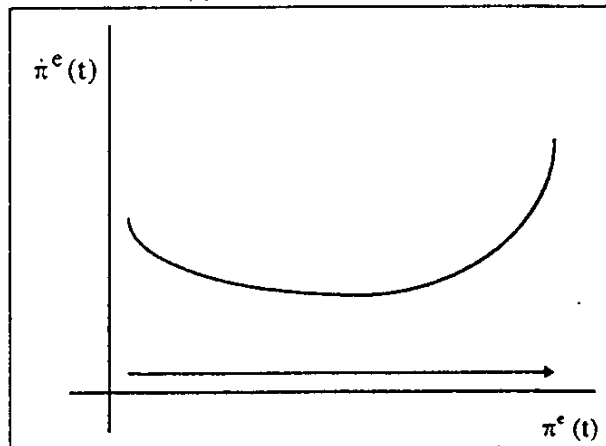
b) The assumption that $G > S^*$ (where S^* represents the maximum steady-state value of seignorage) is equivalent to $G > \pi^e m$ for all possible values of π^e . Thus since $\beta b < 1$, the right-hand side of equation (10) is everywhere positive: regardless of where it starts, expected inflation grows without bound. To examine the shape of the phase diagram, substitute $m(t) = Ce^{-b\pi^e(t)}$ into equation (10):

$$(11) \quad \dot{\pi}^e(t) = \frac{\beta G}{(1 - \beta b)Ce^{-b\pi^e(t)}} - \frac{\beta}{1 - \beta b}\pi^e(t)$$

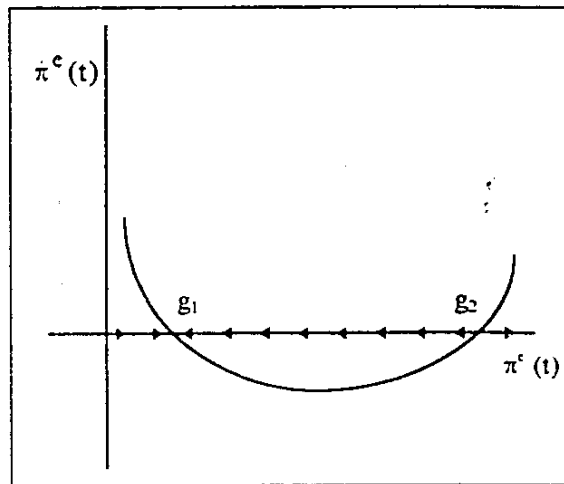
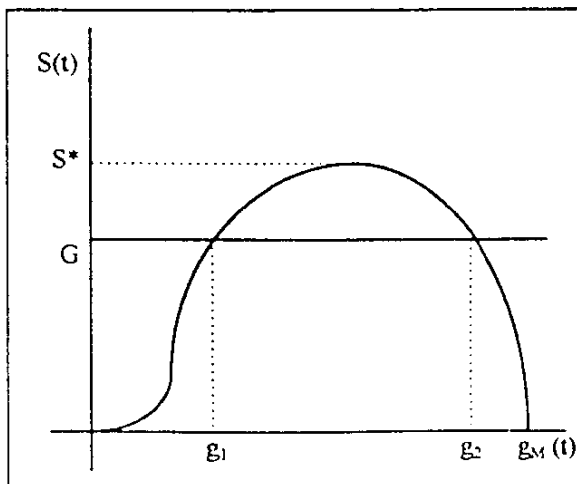
The following derivatives will be useful:

$$(12) \quad \frac{d\dot{\pi}^e(t)}{d\pi^e(t)} = \frac{\beta b G e^{b\pi^e(t)}}{(1 - \beta b)C} - \frac{\beta}{1 - \beta b} \quad \text{and} \quad (13) \quad \frac{d^2\dot{\pi}^e(t)}{d\pi^e(t)^2} = \frac{\beta b^2 G e^{b\pi^e(t)}}{(1 - \beta b)C} > 0$$

By setting the right-hand side of equation (12) equal to zero, it is straightforward to show that $\dot{\pi}^e(t)$ reaches a minimum at $\pi^e(t) = [\ln(C/bG)]/b$. Thus the phase diagram has the shape depicted at right. From equation (1), and since π^e rises without bound, the real money stock is continually falling. If $m(t)$ is continually falling, then – from equation (3) – it must be the case that the growth rate of the nominal money supply, $g_M(t)$, is continually rising if the government is to obtain G in seignorage.



c) Now consider the case of $G < S^*$. The diagram on the left below reproduces Figure 9.8 from the text. It gives the amount of seignorage the government can obtain in steady state as a function of the growth rate of the nominal money supply. In the case of $G < S^*$, there are 2 possible growth rates of the nominal money supply, labeled g_1 and g_2 in the diagram, consistent with raising the amount G in seignorage. Recall that in a steady state, expected inflation equals actual inflation which in turn equals the constant growth rate of the nominal money supply. Thus, by assumption, $\pi^e(t)m(t) = G$ at $\pi^e(t) = g_1$ and $\pi^e(t) = g_2$. From equation (10) then, $\dot{\pi}^e(t) = 0$ at $\pi^e(t) = g_1$ and $\pi^e(t) = g_2$. From the diagram on the left, when $g_1 < \pi^e(t) < g_2$, we have $\pi^e(t)m(t) > G$ and thus $\dot{\pi}^e(t) < 0$. Otherwise, $\pi^e(t)m(t) < G$ and thus $\dot{\pi}^e(t) > 0$. Putting all of this information together gives us the phase diagram depicted on the right. The low-inflation steady state with $\pi^e(t) = \pi(t) = g_1$ is stable and the high-inflation steady state with $\pi^e(t) = \pi(t) = g_2$ is unstable.



SOLUTIONS TO CHAPTER 10

Problem 10.1

a) Substituting the expression for the average wage — $w_a = fw_u + (1 - f)w_n$ — into the expression for the nonunion wage — $w_n = (1 - bu)w_a / (1 - \beta)$ — yields:

$$(1) \quad w_n = \frac{(1 - bu)}{(1 - \beta)} \cdot [fw_u + (1 - f)w_n]$$

Substituting the union wage — $w_u = (1 + \mu)w_n$ — into equation (1) yields:

$$(2) \quad w_n = \frac{(1 - bu)}{(1 - \beta)} \cdot [f(1 + \mu)w_n + (1 - f)w_n] = \frac{(1 - bu)}{(1 - \beta)} \cdot [(1 + \mu f)w_n]$$

Simplifying gives us:

$$(3) \quad (1 - bu)(1 + \mu f) = (1 - \beta)$$

Since $(1 - bu)(1 + \mu f) = 1 + \mu f - bu - b\mu fu$, equation (3) can be rewritten as:

$$(4) \quad -u(b + b\mu f) = -\beta - \mu f$$

and thus the equilibrium unemployment rate is:

$$(5) \quad u = \frac{\beta + \mu f}{b(1 + \mu f)}$$

b) i) Substituting $\mu = f = 0.15$, $\beta = 0.06$ and $b = 1$ into equation (5) gives us:

$$(6) \quad u = \frac{(0.06) + (0.15)(0.15)}{1 + (0.15)(0.15)} = \frac{0.0825}{1.0225} = 0.081$$

Equilibrium unemployment is approximately 8.1%, which is higher than the 6% obtained with $\beta = 0.06$ and $b = 1$ in the standard version of this model without a union sector.

In order to determine by what proportion the cost of effective labor in the union sector exceeds that in the nonunion sector, we need to calculate the equilibrium effort level in each sector. The union wage as a function of the average wage is:

$$(7) \quad w_u = (1 + \mu)w_n = (1 + \mu)(1 - bu)w_a / (1 - \beta)$$

Substituting equation (7) and the definition of the index of labor market conditions — $x = (1 - bu)w_a$ — into the expression for effort, $e = [(w - x)/x]^\beta$ — gives us:

$$(8) \quad e_u = \left[\frac{[(1 + \mu)(1 - bu)w_a / (1 - \beta)] - (1 - bu)w_a}{(1 - bu)w_a} \right]^\beta = \left[\frac{(1 + \mu)(1 - bu) - (1 - \beta)(1 - bu)}{(1 - \beta)(1 - bu)} \right]^\beta$$

or simply:

$$(9) \quad e_u = \left[\left(\frac{1 + \mu}{1 - \beta} \right) - 1 \right]^\beta = \left(\frac{\mu + \beta}{1 - \beta} \right)^\beta$$

Substituting $w_a = (1 - bu)w_u / (1 - \beta)$ into the expression for effort yields:

$$(10) \quad e_n = \left[\frac{[(1 - bu)w_a / (1 - \beta)] - (1 - bu)w_a}{(1 - bu)w_a} \right]^\beta = \left[\frac{(1 - bu) - (1 - \beta)(1 - bu)}{(1 - \beta)(1 - bu)} \right]^\beta$$

or simply:

$$(11) \quad e_n = \left[\frac{1}{1 - \beta} - 1 \right]^\beta = \left(\frac{\beta}{1 - \beta} \right)^\beta$$

In the union sector, it costs a firm w_u to buy 1 unit of labor, which provides e_u units of effective labor. Thus it costs a firm w_u/e_u to buy 1 unit of effective labor. Using the fact that $w_u = (1 + \mu)w_n$ and equation (9), we can write:

$$(12) \quad \frac{w_u}{e_u} = \frac{(1 + \mu)w_n}{[(\mu + \beta)/(1 - \beta)]^\beta}$$

Similarly, the cost to a nonunion firm of obtaining 1 unit of effective labor is w_n/e_n . Using equation (11), we can write:

$$(13) \quad \frac{w_n}{e_n} = \frac{w_n}{[\beta/(1 - \beta)]^\beta}$$

Dividing equation (12) by equation (13) gives us the ratio of the cost of effective labor in the union to the nonunion sector:

$$(14) \quad \frac{w_u/e_u}{w_n/e_n} = \frac{(1 + \mu)w_n}{[(\mu + \beta)/(1 - \beta)]^\beta} \cdot \frac{[\beta/(1 - \beta)]^\beta}{w_n} = (1 + \mu) \cdot \left(\frac{\beta}{\mu + \beta} \right)^\beta$$

Substituting $\mu = 0.15$ and $\beta = 0.06$ into equation (14) gives us:

$$(15) \quad \frac{w_u/e_u}{w_n/e_n} = (1.15) \cdot \left(\frac{0.06}{0.21} \right)^{0.06} = 1.0667$$

Note that although the cost of labor in the union sector exceeds the cost of labor in the nonunion sector by a factor of $(1 + \mu) = 1.15$, the cost of effective labor is only higher by a factor of about 1.07. This is because union workers exert more effort since they are paid a higher wage.

b) i) Substituting $\mu = f = 0.15$, $\beta = 0.03$ and $b = 0.5$ into equation (5) yields:

$$(16) \quad u = \frac{(0.03) + (0.15)(0.15)}{0.5[1 + (0.15)(0.15)]} = \frac{0.0525}{0.51125} = 0.103$$

Equilibrium unemployment is now higher at about 10.3%. Substituting $\mu = 0.15$ and $\beta = 0.03$ into equation (14) gives us:

$$(17) \quad \frac{w_u/e_u}{w_n/e_n} = (1.15) \cdot \left(\frac{0.03}{0.18} \right)^{0.03} = 1.0898$$

With the elasticity of effort with respect to the wage lower at $\beta = 0.03$ and less weight on unemployment in the index of labor market conditions, the ratio of the cost of effective labor in the union sector to that in the nonunion sector is now higher.

Problem 10.2

a) i) With $e = 1$ and taking w as given, the firm's problem is to choose L in order to maximize profits as given by:

$$(1) \quad \pi = L^\alpha / \alpha - wL$$

The first-order condition is:

$$(2) \quad \partial\pi/\partial L = L^{\alpha-1} - w = 0$$

and thus the firm's choice of employment is:

$$(3) \quad L = w^{-1/(1-\alpha)}$$

Substituting equation (3) into the expression for profits yields:

$$\pi = w^{-1/(1-\alpha)} / \alpha - w^{(1-\alpha)/(1-\alpha)} = w^{-\alpha/(1-\alpha)} [(1/\alpha) - 1]$$

and thus the level of profits is:

$$(4) \quad \pi = [(1 - \alpha)/\alpha] w^{-\alpha/(1-\alpha)}$$

a) ii) Substituting $U = w - x$ and equation (4) for profits into $U^\gamma \pi^{1-\gamma}$, the bargaining problem becomes:

$$\max_w (w - x)^\gamma \left[\left(\frac{1 - \alpha}{\alpha} \right) w^{-\alpha/(1-\alpha)} \right]^{1-\gamma}$$

It will simplify the algebra to maximize the log of $U^\gamma \pi^{1-\gamma}$ so the problem becomes:

$$(5) \max_w \gamma \ln(w - x) + (1 - \gamma) \ln \left(\frac{1 - \alpha}{\alpha} \right) - \frac{(1 - \gamma)\alpha}{1 - \alpha} \ln w$$

The first-order condition is:

$$(6) \frac{\partial [\ln(U^\gamma \pi^{1-\gamma})]}{\partial w} = \frac{\gamma}{w - x} - \frac{(1 - \gamma)\alpha}{(1 - \alpha)} \cdot \frac{1}{w} = 0$$

Equation (6) can be written as:

$$(7) \frac{\gamma}{w - x} = \frac{(1 - \gamma)\alpha}{(1 - \alpha)} \cdot \frac{1}{w} \Rightarrow (1 - \alpha)\gamma w = (1 - \gamma)\alpha w - (1 - \gamma)\alpha x$$

Simplifying yields:

$$(8) [\gamma - \alpha\gamma - \alpha + \alpha\gamma]w = -(1 - \gamma)\alpha x$$

and thus finally, the wage chosen in the bargaining process is:

$$(9) w = \frac{(1 - \gamma)\alpha x}{(\alpha - \gamma)}$$

a) iii) Taking the log of both sides of equation (9) yields:

$$(10) \ln w = \ln(1 - \gamma) + \ln(\alpha x) - \ln(\alpha - \gamma)$$

The derivative of the log wage with respect to the union's bargaining power is:

$$(11) \frac{\partial \ln w}{\partial \gamma} = \frac{-1}{1 - \gamma} + \frac{1}{\alpha - \gamma}$$

Evaluating this derivative at $\gamma = 0$ gives us:

$$(12) \left. \frac{\partial \ln w}{\partial \gamma} \right|_{\gamma=0} = -1 + \frac{1}{\alpha} = \frac{1 - \alpha}{\alpha}$$

b) i) Substituting $e = [(w - x)/x]^\beta$ into the expression for profits allows us to write the firm's problem as:

$$(13) \max_L \pi = \left(\frac{1}{\alpha} \right) \cdot \left(\frac{w - x}{x} \right)^{\alpha\beta} L^\alpha - wL$$

The first-order condition is:

$$(14) \frac{\partial \pi}{\partial L} = \left(\frac{w - x}{x} \right)^{\alpha\beta} L^{\alpha-1} - w = 0$$

and thus the firm's choice of employment is:

$$(15) L = \left(\frac{w - x}{x} \right)^{\alpha\beta/(1-\alpha)} w^{-1/(1-\alpha)}$$

Substituting equation (15) into the expression for profits yields:

$$(16) \pi = \left(\frac{1}{\alpha} \right) \cdot \left(\frac{w - x}{x} \right)^{\alpha\beta} \left(\frac{w - x}{x} \right)^{\alpha^2\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)} - w^{1-[1/(1-\alpha)]} \left(\frac{w - x}{x} \right)^{\alpha\beta/(1-\alpha)}$$

Since $\alpha\beta + [\alpha^2\beta/(1-\alpha)] = [\alpha\beta - \alpha^2\beta + \alpha^2\beta]/(1-\alpha) = \alpha\beta/(1-\alpha)$ and $1 - [1/(1-\alpha)] = [(1-\alpha-1)/(1-\alpha)] = -\alpha/(1-\alpha)$, equation (16) can be written as:

$$(17) \pi = \left(\frac{1}{\alpha}\right) \cdot \left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)} - w^{-\alpha/(1-\alpha)} \left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)}$$

Collecting terms and simplifying yields:

$$(18) \pi = \left(\frac{1-\alpha}{\alpha}\right) \cdot \left(\frac{w-x}{x}\right)^{\alpha\beta/(1-\alpha)} w^{-\alpha/(1-\alpha)}$$

b) ii) We will again maximize the log of $U^\gamma \pi^{1-\gamma}$. Substituting $U = w - x$ and equation (18) for profits into $\ln[U^\gamma \pi^{1-\gamma}]$ – and ignoring the terms that do not depend upon w – we have the bargaining problem:

$$(19) \max_w \gamma \ln(w-x) + \frac{(1-\gamma)\alpha\beta}{(1-\alpha)} \ln(w-x) - \frac{\alpha(1-\gamma)}{(1-\alpha)} \ln w$$

The first-order condition is:

$$(20) \frac{\partial [\ln(U^\gamma \pi^{1-\gamma})]}{\partial w} = \frac{\gamma}{w-x} + \frac{(1-\gamma)\alpha\beta}{(1-\alpha)(w-x)} - \frac{\alpha(1-\gamma)}{(1-\alpha)w} = 0$$

Equation (20) becomes:

$$\frac{(1-\alpha)\gamma + (1-\gamma)\alpha\beta}{(1-\alpha)(w-x)} = \frac{\alpha(1-\gamma)}{(1-\alpha)w} \Rightarrow w[(1-\alpha)\gamma + (1-\gamma)\alpha\beta] = \alpha(1-\gamma)(w-x)$$

Collecting the terms in w , we have:

$$(21) w[(1-\alpha)\gamma + (1-\gamma)\alpha\beta - \alpha(1-\gamma)] = -\alpha(1-\gamma)x$$

Since $(1-\alpha)\gamma + (1-\gamma)\alpha\beta - \alpha(1-\gamma) = \gamma - \alpha\gamma + (1-\gamma)\alpha\beta - \alpha + \alpha\gamma = -\alpha + (1-\gamma)\alpha\beta$, solving for w yields:

$$(22) w = \frac{\alpha(1-\gamma)x}{\alpha - \gamma - (1-\gamma)\alpha\beta}$$

Note that equation (22) does reduce to equation (9) if $\beta = 0$.

b) iii) Taking the log of both sides of equation (22) yields:

$$(23) \ln w = \ln(\alpha x) + \ln(1-\gamma) - \ln[\alpha - \gamma - (1-\gamma)\alpha\beta]$$

The derivative of the log wage with respect to the union's bargaining power is:

$$(24) \frac{\partial \ln w}{\partial \gamma} = \frac{-1}{1-\gamma} + \frac{1}{\alpha - \gamma - (1-\gamma)\alpha\beta} (-1 + \alpha\beta)$$

Evaluating this derivative at $\gamma = 0$ gives us:

$$(25) \left. \frac{\partial \ln w}{\partial \gamma} \right|_{\gamma=0} = -1 - \frac{1}{\alpha - \alpha\beta} (\alpha\beta - 1) = \frac{-\alpha + \alpha\beta - \alpha\beta + 1}{\alpha(1-\beta)}$$

or simply:

$$(26) \left. \frac{\partial \ln w}{\partial \gamma} \right|_{\gamma=0} = \frac{1-\alpha}{\alpha(1-\beta)}$$

Comparing equation (26) with equation (12), it is true – since $0 < \beta < 1$ – that the elasticity is higher with efficiency wages. But the qualitative effect is the same in either case. A marginal increase in the union's bargaining power increases the wage whether or not there are efficiency wages – it just increases more in the presence of efficiency wages.

Problem 10.3

The no-shirking condition (NSC) is given by:

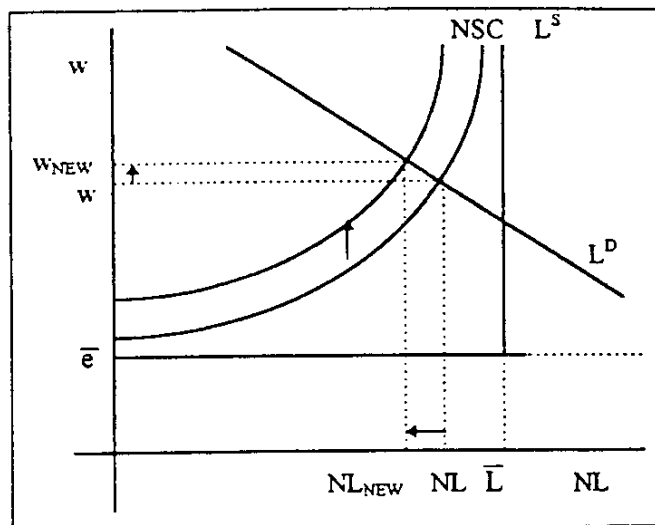
$$(1) \quad w = \bar{e} + \left(\rho + \frac{\bar{L}}{\bar{L} - NL} b \right) \frac{\bar{e}}{q}$$

and the labor demand curve is given by:

$$(2) \quad F'(\bar{e}L) = w/\bar{e}$$

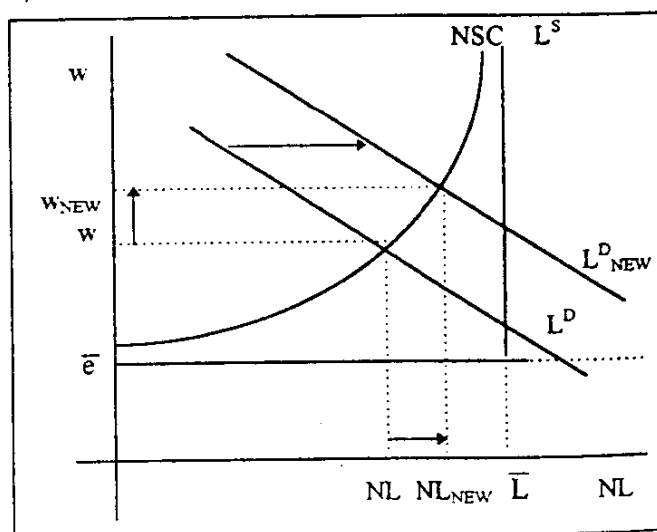
Equation (2) states that firms choose L so that the marginal product of effective labor equals the marginal cost of effective labor, where the wage (w) is set to satisfy (1).

a) An increase in ρ shifts up the no-shirking locus. From equation (1), for a given NL , the wage needed to get workers to exert effort is now higher. Intuitively, since workers discount the future more, it matters less to them if they are caught shirking, are fired and have to go through a period of unemployment. Thus at a given level of employment, firms must pay a higher wage to deter shirking. The labor demand curve is unaffected. As the diagram at right depicts, equilibrium employment falls and the equilibrium wage rises.

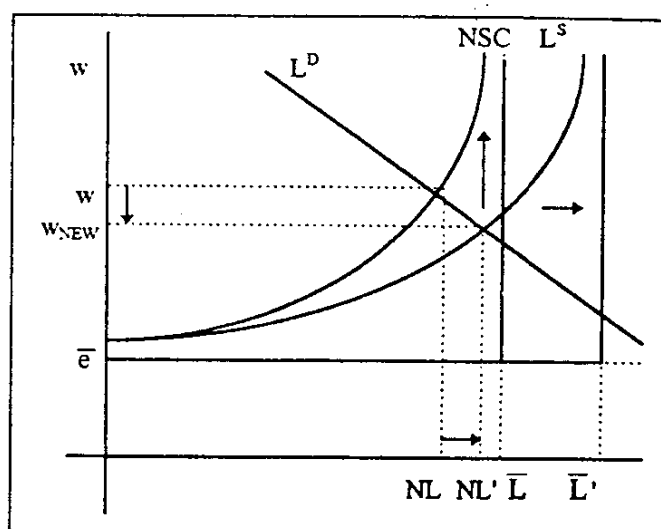


b) An increase in the job breakup rate, b , shifts up the no-shirking locus. From equation (1), for a given NL , the wage required to get workers to exert effort is now higher. Intuitively, since workers are more likely to lose their job anyway, the value of being employed is lower. Thus workers are not as concerned about being caught shirking and fired. So at a given level of employment, firms must pay a higher wage to deter shirking. The labor demand curve is unaffected. Equilibrium employment falls and the wage rises.

c) The rise in A shifts the labor demand curve to the right. The no-shirking locus is unaffected. As the diagram at right depicts, the equilibrium wage rises as does the level of employment. Note that if efficiency wages were not present, inelastic labor supply would mean that increases in technology would lead only to increases in the wage, not to increases in employment.



d) The vertical portion of the labor supply curve shifts to the right. The labor demand curve is unaffected. The no-shirking locus shifts down. Intuitively, at a given NL, $\bar{L} - NL$ is now higher. Thus at a given level of employment, if workers become unemployed, they are likely to stay unemployed longer. Thus at a given level of employment, the cost of shirking is greater for a worker and thus firms can get away with paying a lower wage to deter shirking. From the diagram at right, the equilibrium wage falls and employment rises.



Problem 10.4

a) The total number of unemployed workers is $\bar{L} - NL$. If there is no shirking, the number of workers becoming unemployed per unit time is the number of firms, N , times the number of workers per firm, L , times the rate of job breakup, b . In a steady state, this is also the number of workers becoming employed per unit time. If people who have been unemployed the longest are hired first, the length of time it takes to get a job, which we can denote t^* , is equal to the total number of unemployed workers divided by the number of people who get hired per unit time. For example, if there were 1000 unemployed workers and 100 people become employed per unit time, then the length of time it takes to get a job is $1000/100 = 10$. Thus in general:

$$(1) \quad t^* = \frac{\bar{L} - NL}{NLb}$$

b) There is no uncertainty involved when calculating the value of becoming newly unemployed as a function of the value of being employed. When a worker loses her job, she knows that she will be unemployed for $t^* = (\bar{L} - NL)/NLb$ units of time, at which point she will become employed again. Thus the value of becoming newly unemployed is that t^* units of time into the future, the individual will have the value of being employed. The discounted value of being employed t^* units of time into the future is given by $e^{-\rho t^*} V_E$. Thus:

$$(2) \quad V_U = e^{-\rho(\bar{L}-NL)/NLb} V_E$$

c) As in the usual version of the Shapiro-Stiglitz model, the firm chooses a wage so that the value of being employed, V_E , just equals the value of shirking, V_S . From equation (10.35) in the text, this implies:

$$(3) \quad V_E - V_U = \frac{\bar{e}}{q}$$

Substituting equation (2) into equation (3) yields:

$$V_E - e^{-\rho t^*} V_E = \frac{\bar{e}}{q}$$

Solving for V_E gives us:

$$(4) \quad V_E = \frac{\bar{e}}{(1 - e^{-\rho t^*})q}$$

The next step is to determine what the wage must be in order for the value of employment to be given by equation (4). From equation (10.30) in the text for the return from being employed -- $\rho V_E = (w - \bar{e}) - b(V_E - V_U)$ -- we can solve for w :

$$(5) \quad w = \bar{e} + \rho V_E + b(V_E - V_U)$$

Substituting equations (3) and (4) into equation (5) yields:

$$(6) \quad w = \bar{e} + \frac{\rho \bar{e}}{(1 - e^{-\rho t^*})q} + \frac{b\bar{e}}{q}$$

Substituting $t^* = (\bar{L} - NL)/NLb$ into equation (6) gives us:

$$(7) \quad w = \bar{e} + \left[\frac{\rho}{1 - e^{-\rho(\bar{L} - NL)/NLb}} + b \right] \frac{\bar{e}}{q}$$

Equation (7) is the no-shirking condition. Note that as $NL \rightarrow \bar{L}$ (as unemployment goes to zero), there is no wage that will deter shirking. This is because as the number of unemployed workers goes to zero, there is no line to stand in and wait for a job. An individual who is caught shirking and is fired will be at the front of the line and will be rehired instantly. As $NL \rightarrow 0$, the wage needed to deter shirking goes to $\bar{e} + (\rho + b)\bar{e}/q$. This is exactly the same wage needed to deter shirking as $NL \rightarrow 0$ in the standard Shapiro-Stiglitz model.

d) In order to compare the equilibrium unemployment rate in this model with the equilibrium unemployment rate in the Shapiro-Stiglitz model, we need to compare the two no-shirking loci. If the wage needed to deter shirking for a given level of employment is higher in one of the two models, equilibrium unemployment will be higher in that model.

Intuitively, in both models, the value of being newly unemployed comes from the possibility of becoming employed. For a given level of employment, the expected time to becoming employed is the same in the two models. Here it is certain; in the Shapiro-Stiglitz model, it is uncertain. That is, in this model, the newly unemployed worker knows that she will be rehired in t^* units of time; in the Shapiro-Stiglitz model, she has probability $1/t^*$ per unit time of becoming employed again and thus on average will be employed again in t^* units of time.

Now, since $e^{-\rho t}$ is convex in t , the uncertainty about the time it takes to get employed again in the Shapiro-Stiglitz model raises V_U for a given V_E relative to this model. This means that firms must pay a higher wage in the Shapiro-Stiglitz model, for a given level of employment, to deter shirking. Thus equilibrium unemployment is higher in the Shapiro-Stiglitz model.

More formally, our claim is that:

(8) NSC wage for a given NL in Shapiro-Stiglitz > NSC wage for a given NL in this model

From equation (10.39) in the text and equation (7) here, the claim is:

$$(9) \quad \bar{e} + \left[\rho + \frac{\bar{L}}{\bar{L} - NL} b \right] \frac{\bar{e}}{q} > \bar{e} + \left[\frac{\rho}{1 - e^{-\rho t^*}} + b \right] \frac{\bar{e}}{q}$$

Subtracting $\bar{e}$ from both sides and dividing both sides of the resulting expression by $\bar{e}/q$ leaves us with:

$$(10) \quad \rho + \frac{\bar{L}}{\bar{L} - NL} b > \frac{\rho}{1 - e^{-\rho t^*}} + b$$

Now, using the definition of $t^* = \frac{\bar{L} - NL}{NLb}$, we can write $NLbt^* = \bar{L} - NL$ or:

$$(11) \quad NL = \bar{L}/(1 + bt^*)$$

Substituting equation (11) into $[\bar{L}/(\bar{L} - NL)]b$ gives us:

$$(12) \quad \frac{\bar{L}}{\bar{L} - NL} b = \frac{\bar{L}}{\bar{L} - [\bar{L}/(1 + bt^*)]} b = \frac{1 + bt^*}{(1 + bt^*) - 1} b = \frac{1 + bt^*}{t^*}$$

Substituting equation (12) into our claim gives us:

$$(13) \quad \rho + \frac{1 + bt^*}{t^*} > \frac{\rho}{1 - e^{-\rho t^*}} + b$$

Multiplying both sides of (13) by t^* gives us an equivalent expression:

$$(14) \quad \rho t^* + 1 + bt^* > \frac{\rho t^*}{1 - e^{-\rho t^*}} + bt^*$$

Subtracting bt^* from both sides of (14) and then multiplying both sides of the resulting expression by $(1 - e^{-\rho t^*})$ yields:

$$(15) \quad \rho t^* - \rho t^* e^{-\rho t^*} + 1 - e^{-\rho t^*} > \rho t^*$$

Thus finally, our original claim is equivalent to:

$$(16) \quad 1 - e^{-\rho t^*} - \rho t^* e^{-\rho t^*} > 0$$

We need to show that (16) actually holds. Note that it takes the form $1 - e^{-x} - xe^{-x}$ where $x = \rho t^*$. This expression is greater than zero for $x > 0$. To formally see this, let $f(x)$ denote the left-hand side of equation (16). Then $f(0) = 0$ and:

$$(17) \quad f'(x) = e^{-x} + xe^{-x} - e^{-x} = xe^{-x} > 0 \text{ for } x > 0$$

Thus since f is equal to zero at zero, and is increasing for all x greater than zero, it must be positive for all $x > 0$. Therefore, our claim holds and equilibrium unemployment is higher in the Shapiro-Stiglitz model than it is in this model.

Problem 10.5

a) As explained in Section 10.2 of the text, the firm maximizes profit by picking a wage such that the elasticity of effort with respect to the wage is equal to one, or:

$$(1) \quad [w/e][\partial e/\partial w] = 1$$

Note that for $w < w^*$, $\partial e/\partial w = 1/w^*$ and $e = w/w^*$. Thus equation (1) is satisfied for any wage up to (and essentially including) the fair wage, w^* . Thus we can assume that the firm pays the fair wage, w^* .

Another way of thinking about this is to utilize another result from the firm's problem which is that they choose the wage so that the cost of a unit of effective labor is minimized. The cost to the firm of a unit of effective labor is w/e .

For $w/w^* > 1$, we have:

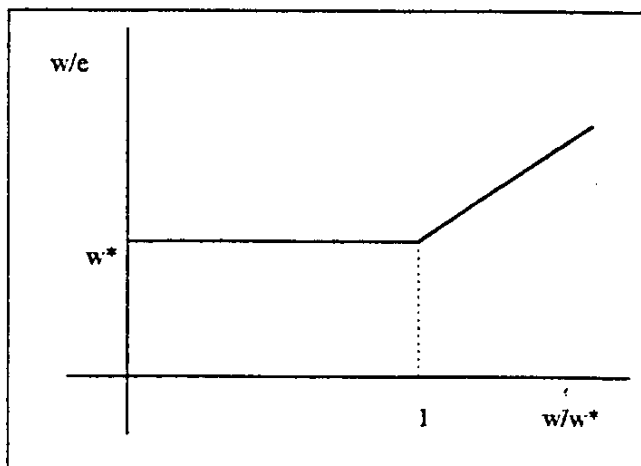
$$(2) \quad \frac{w}{e} = \frac{w}{w/w^*} = w^*$$

For $w/w^* > 1$, we have:

$$(3) \quad \frac{w}{e} = \frac{w}{1} = w$$

See the diagram at right, which plots the cost of a unit of effective labor as a function of the firm's wage relative to the fair wage. We can

see that any wage such that $w/w^* \leq 1$ or $w \leq w^*$, minimizes the cost of effective labor. Given the assumption that the firm pays the highest wage in this range, it chooses $w = w^*$.



b) i) As explained in part (a), if the firm can choose the wage freely, it pays the fair wage. Thus:

$$(4) \quad w = \bar{w} + a - bu$$

b) ii) Assume that there is positive unemployment. If this is the case, the firm is unconstrained in its choice of the wage and therefore pays the fair wage, nothing more. Since this is true for all firms, the average wage, $\bar{w}$, must equal w . Hence:

$$(5) \quad w = w + a - bu$$

Solving equation (5) for the unemployment rate yields:

$$(6) \quad u = a/b$$

Note that the higher is a – the higher above the average wage is the perceived fair wage – the higher is the equilibrium unemployment rate. Also, the lower is b – the less responsive is the fair wage to unemployment – the higher is the equilibrium unemployment rate.

Since we have derived this under the assumption of positive unemployment, we need to verify that this is actually the case. If $u = 0$, then the fair wage is $w^* = w + a$, where we have used the fact that $\bar{w} = w$ (all firms pay the same wage). But if $a > 0$, this means that $w < w^*$ – firms are paying less than the fair wage. But this violates our assumption that firms do not choose to pay below the fair wage. So as long as $a > 0$, there will be positive unemployment in equilibrium.

b) iii) From part (b), (ii), we can see that if $a \leq 0$, equilibrium unemployment will be zero. If $a = 0$, the fair wage will simply be equal to the wage which in turn equals the marginal product of the full-employment labor force. If $a < 0$, the perceived fair wage is always less than the average wage for any value of unemployment. Thus the representative firm, taking the average wage as given, wants to pay less than the average wage. Since workers are willing to work at any positive wage, firms need only pay ϵ above 0 to get workers to be willing to work, even with zero unemployment.

c) i) The first point is that the high-wage workers cannot have unemployment. Suppose the Type 1 workers have a higher wage. Then w_1 is higher than the average wage or:

$$(7) \quad w_1 > (w_1 + w_2)/2$$

where we have used the fact that all firms pay the same wage so that $\bar{w}_1 = w_1$ and $\bar{w}_2 = w_2$. Thus the fair wage for Type 1 workers is:

$$(8) \quad w_1^* = (w_1 + w_2)/2 - bu_1 < w_1$$

But inequality (8) says that the firm is paying a wage higher than the fair wage to a group that has unemployment. But the firm does not need to do this – it is unconstrained in its choice of w_1 if $u_1 > 0$. It could cut the wage down to the fair wage level. Thus there cannot be unemployment among the high-wage workers.

The next point is that the high-productivity workers have no unemployment. Why? Suppose they did. Then from what was just discussed, they could not be the high-wage group, since the high-wage group has no unemployment. This would mean the high-productivity workers are paid a lower wage than the low-productivity workers (who must then be fully employed). But this is not possible, since at a given wage the firm would rather hire high-productivity workers since their marginal product is higher. Thus there cannot be unemployment among the high-productivity workers.

c) ii) In equilibrium, there will be unemployment among low-productivity workers. In part (c), (i), we have explained why the high-productivity workers are the high-wage workers. Thus the low-productivity workers must be the low-wage workers. This means that their wage is less than the average wage, or:

$$(9) \quad w_2 < (w_1 + w_2)/2$$

Now suppose that there was no unemployment among the low-productivity workers. Then the fair wage for them would be:

$$(10) \quad w_2^* = (w_1 + w_2)/2 > w_2$$

But inequality (10) cannot be true – the firm will not pay a wage below the fair wage. Thus there must be some positive unemployment rate $u_2 > 0$ such that $w_2 = w_2^*$.

Problem 10.6

a) Suppose there are N states of the world. Then the firm's expected profits are:

$$(1) \quad E(\pi) = \sum_{i=1}^N p_i [A_i F(L_i) - C_i^E L_i - C_i^U (\bar{L} - L_i)]$$

The expected utility of a representative worker is:

$$(2) \quad E(u) = \sum_{i=1}^N p_i \left\{ \left(\frac{L_i}{\bar{L}} \right) [U(C_i^E) - K] + \left(\frac{\bar{L} - L_i}{\bar{L}} \right) U(C_i^U) \right\}$$

The firm's problem is to choose the L_i 's, C_i^E 's and C_i^U 's in order to maximize equation (1) subject to equation (2). Thus the Lagrangian is:

$$\mathcal{L} = \sum_{i=1}^N p_i [A_i F(L_i) - C_i^E L_i - C_i^U (\bar{L} - L_i)] + \lambda \left[\sum_{i=1}^N p_i \left\{ \left(\frac{L_i}{\bar{L}} \right) [U(C_i^E) - K] + \left(\frac{\bar{L} - L_i}{\bar{L}} \right) U(C_i^U) \right\} - u_0 \right]$$

b) The first-order conditions are:

$$(3) \quad \frac{\partial \mathcal{L}}{\partial L_i} = p_i A_i F'(L_i) - p_i C_i^E + p_i C_i^U + \lambda p_i \left(\frac{1}{\bar{L}} \right) [U(C_i^E) - K] - \lambda p_i \left(\frac{1}{\bar{L}} \right) U(C_i^U) = 0$$

$$(4) \quad \frac{\partial \mathcal{L}}{\partial C_i^E} = -p_i L_i + \lambda p_i \left(\frac{L_i}{\bar{L}} \right) U'(C_i^E) = 0$$

$$(5) \quad \frac{\partial \mathcal{L}}{\partial C_i^U} = -p_i (\bar{L} - L_i) + \lambda p_i \left(\frac{\bar{L} - L_i}{\bar{L}} \right) U'(C_i^U) = 0$$

Solving equation (4) for $U'(C_i^E)$ gives us:

$$(6) \quad U'(C_i^E) = \bar{L}/\lambda$$

Equation (6) implies the marginal utility of consumption for the employed workers is constant across states and thus – with $U''(\bullet) < 0$ – consumption of the employed workers is constant across states.

Solving equation (5) for $U'(C_i^U)$ gives us:

$$(7) \quad U'(C_i^U) = \bar{L}/\lambda$$

Thus consumption of the unemployed workers is also constant across states. Comparing equations (6) and (7), it is also true that the marginal utility of consumption is the same for both employed and unemployed workers. This implies that the level of consumption of both types of workers is the same. That is:

$$(8) \quad C^E = C^U$$

So C^E and C^U do not depend on the state and are always equal to each other.

c) The unemployed workers are actually better off. They consume the same amount as the employed workers and do not suffer the disutility of work, K .

Problem 10.7

a) We can maximize $2E(\pi) = A_G F(L_G) - wL_G + A_B F(L_B) - wL_B - fB(L_G - L_B)$ subject to $(w - K) + (L_B/L_G)(w - K) + [(L_G - L_B)/L_G]B = 2u_0$. Thus the Lagrangian is:

$$(1) \quad \mathcal{L} = A_G F(L_G) - wL_G + A_B F(L_B) - wL_B - fB(L_G - L_B) + \lambda \{ (w - K) + (L_B/L_G)(w - K) + [(L_G - L_B)/L_G]B - 2u_0 \}$$

b) The first-order conditions are:

$$(2) \partial \mathcal{L} / \partial w = -L_G - L_B + \lambda + \lambda(L_B / L_G) = 0$$

$$(3) \partial \mathcal{L} / \partial L_B = A_B F'(L_B) - w + fB + (\lambda / L_G)(w - K) - (\lambda / L_G)B = 0$$

$$(4) \partial \mathcal{L} / \partial L_G = A_G F'(L_G) - w - fB - (\lambda L_B / L_G^2)(w - K) + (\lambda L_B / L_G^2)B = 0$$

c) We can solve equation (2) for λ :

$$\lambda[(L_G + L_B) / L_G] = L_G + L_B$$

or simply:

$$(5) \lambda = L_G$$

Substituting equation (5) into equation (3) yields:

$$A_B F'(L_B) - w + fB + (w - K) - B = 0$$

which implies:

$$(6) A_B F'(L_B) - K - B(1 - f) = 0$$

Differentiating both sides of equation (6) with respect to f gives us:

$$(7) A_B F''(L_B) [\partial L_B / \partial f] + B = 0$$

Solving equation (7) for $\partial L_B / \partial f$ yields:

$$(8) \frac{\partial L_B}{\partial f} = \frac{-B}{A_B F''(L_B)} > 0$$

$\partial L_B / \partial f > 0$ since $B > 0$, $A_B > 0$ and $F''(\bullet) < 0$. Thus a fall in the fraction of the unemployment benefit paid by the firm actually causes the firm to hire fewer workers in the bad state.

Similarly, we can differentiate both sides of equation (6) by B to yield:

$$(9) A_B F''(L_B) [\partial L_B / \partial B] - (1 - f) = 0$$

Solving equation (9) for $\partial L_B / \partial B$ gives us:

$$(10) \frac{\partial L_B}{\partial B} = \frac{(1 - f)}{A_B F''(L_B)} < 0 \quad \text{as long as } f < 1$$

Therefore, a rise in the unemployment benefit causes the firm to hire fewer workers in the bad state.

d) Substituting equation (5) into equation (4) yields:

$$A_G F'(L_G) - w - fB - (L_B / L_G)(w - K) + (L_B / L_G)B = 0$$

or simply:

$$(11) A_G F'(L_G) - w - fB - (L_B / L_G)(w - K - B) = 0$$

Equation (11) can be rearranged to obtain:

$$(12) A_G F'(L_G) - fB = w + (L_B / L_G)(w - K - B)$$

Multiplying both sides of the expected utility constraint by 2 yields:

$$(13) w - K + (L_B / L_G)(w - K) + [(L_G - L_B) / L_G]B = 2u_0$$

Equation (13) can be rearranged to obtain:

$$(14) w + (L_B / L_G)(w - K - B) = 2u_0 + K - B$$

Equations (12) and (14) therefore imply:

$$(15) A_G F'(L_G) - fB = 2u_0 + K - B$$

or simply:

$$(16) A_G F'(L_G) = (2u_0 + K) - (1 - f)B$$

Differentiating both sides of equation (16) with respect to f yields:

$$(17) A_G F''(L_G) [\partial L_G / \partial f] = B$$

Solving equation (17) for $\partial L_G / \partial f$ yields:

$$(18) \quad \frac{\partial L_G}{\partial f} = \frac{B}{A_G F''(L_G)} < 0$$

$\partial L_G / \partial f < 0$ since $B > 0$, $A_G > 0$ and $F''(\bullet) < 0$. Thus a fall in the fraction of the unemployment benefit paid by the firm actually causes the firm to hire more workers in the good state.

Similarly, differentiating both sides of equation (16) with respect to B yields:

$$(19) \quad A_G F''(L_G) [\partial L_G / \partial B] = -(1 - f)$$

Solving (19) for $\partial L_G / \partial B$ gives us:

$$(20) \quad \frac{\partial L_G}{\partial B} = \frac{-(1 - f)}{A_G F''(L_G)} > 0 \quad \text{as long as } f < 1$$

Therefore, a rise in the unemployment benefit causes the firm to hire more workers in the good state.

So an increase in unemployment benefits or a reduction in the fraction of those benefits paid by firms will increase employment fluctuations by causing firms to hire less labor in the bad state and more labor in the good state.

Problem 10.8

a) With efficient contracts — as shown in Section 10.5 of the text — $C = wL$ is constant across states. In addition, employment is increasing in A so that $L_G > L_B$. Given A and given the fact that wL is constant, profit — $\pi = AF(L) - wL$ — is increasing in employment. Thus when the true state is A_B , the firm is better off announcing that A is actually A_G and employing L_G . When the true state is A_G , the firm is again better off announcing that A is A_G . Thus when the state is A_G , it is in the firm's interest to announce the true state. However, in the bad state, it is not in the firm's interest to announce the true state and so the efficient contract is not incentive-compatible.

b) The incentive-compatibility constraint that is binding is that the firm not prefer to claim that $A = A_G$ when in fact $A = A_B$. Assuming that this constraint holds with equality, this requires:

$$(1) \quad A_B F(L_B) - C_B = A_B F(L_G) - C_G$$

The left-hand side of equation (1) is the firm's profit in the bad state if it announces the bad state, while the right-hand side of equation (2) is the firm's profit in the bad state if it announces the good state. The other constraint is that workers' expected utility be equal to u_0 or:

$$(2) \quad [U(C_B) - V(L_B)]/2 + [U(C_G) - V(L_G)]/2 = u_0$$

The firm's expected profit is:

$$(3) \quad E(\pi) = [A_B F(L_B) - C_B]/2 + [A_G F(L_G) - C_G]/2$$

The problem facing the firm is to choose L_B , C_B , L_G and C_G to maximize expected profits subject to equations (1) and (2). The Lagrangian is:

$$(4) \quad \mathcal{L} = [A_B F(L_B) - C_B]/2 + [A_G F(L_G) - C_G]/2 + \lambda_1 \{ [U(C_B) - V(L_B)]/2 + [U(C_G) - V(L_G)]/2 - u_0 \} + \lambda_2 \{ [A_B F(L_B) - C_B]/2 - [A_B F(L_G) - C_G] \}$$

The first-order conditions are:

$$(5) \quad \partial \mathcal{L} / \partial C_B = (-1/2) + (1/2)\lambda_1 U'(C_B) - \lambda_2 = 0$$

$$(6) \quad \partial \mathcal{L} / \partial C_G = (-1/2) + (1/2)\lambda_1 U'(C_G) + \lambda_2 = 0$$

$$(7) \quad \partial \mathcal{L} / \partial L_B = (1/2)A_B F'(L_B) - (1/2)\lambda_1 V'(L_B) + \lambda_2 A_B F'(L_B) = 0$$

$$(8) \quad \partial \mathcal{L} / \partial L_G = (1/2)A_G F'(L_G) - (1/2)\lambda_1 V'(L_G) - \lambda_2 A_B F'(L_G) = 0$$

$$(17) \left\{ A_G + \left(\frac{A_G - A_B}{2} \right) \left[\frac{U'(C_B) - U'(C_G)}{U'(C_G)} \right] \right\} F'(L_G) = \frac{V'(L_G)}{U'(C_G)}$$

The contract must clearly involve $L_G > L_B$ and must therefore have $C_G > C_B$ to be incentive-compatible. [Note that if $L_G = L_B$, then C_G must equal C_B for incentive-compatibility. But then equation (10) implies $\lambda_2 = 0$, which implies that equations (7) and (8) cannot both be satisfied.] Since $C_G > C_B$, it follows that $U'(C_B) > U'(C_G)$ and so the second term in brackets is positive. Thus $V'(L_G)/U'(C_G)$ exceeds $A_G F'(L_G)$. The marginal disutility of work exceeds the marginal product of labor in the good state or in other words, there is overemployment in the good state.

e) Given the fact that there is no unemployment in the bad state and overemployment in the good state, this model does not appear to be helpful in understanding the high level of average unemployment. But since it is in the good state that the overemployment occurs, the model does suggest a reason that employment might be procyclical and more responsive to shocks than under symmetric information.

Problem 10.9

a) i) The firm chooses L in order to maximize profits as given by:

$$(1) \pi = AL^\alpha / \alpha - wL$$

The first-order condition is:

$$(2) \partial \pi / \partial L = AL^{\alpha-1} - w = 0 \Rightarrow L^{\alpha-1} = w/A \Rightarrow L = (w/A)^{1/(\alpha-1)}$$

and thus the firm's choice of L is:

$$(3) L = A^{1/(1-\alpha)} w^{-1/(1-\alpha)}$$

a) ii) Substituting equation (3) for the firm's choice of L into the union's objective function yields:

$$U^U = [U(w) - K] A^{1/(1-\alpha)} w^{-1/(1-\alpha)} + U(w_u) [N - A^{1/(1-\alpha)} w^{-1/(1-\alpha)}]$$

Collecting terms gives us:

$$(4) U^U = A^{1/(1-\alpha)} w^{-1/(1-\alpha)} [U(w) - K - U(w_u)] + U(w_u) N$$

The union's problem is to choose w in order to maximize its utility as given in equation (4). The first-order condition is:

$$(5) \partial U^U / \partial w = -[1/(1-\alpha)] A^{1/(1-\alpha)} w^{[-1/(1-\alpha)]-1} [U(w) - K - U(w_u)] + A^{1/(1-\alpha)} w^{-1/(1-\alpha)} [U'(w)] = 0$$

Equation (5) can be written as:

$$[1/(1-\alpha)] A^{1/(1-\alpha)} w^{[-1/(1-\alpha)]-1} [U(w) - K - U(w_u)] = A^{1/(1-\alpha)} w^{-1/(1-\alpha)} U'(w)$$

which simplifies to:

$$[1/(1-\alpha)] w^{-1} [U(w) - K - U(w_u)] = U'(w)$$

and thus w is defined implicitly by:

$$(6) w = \left(\frac{1}{1-\alpha} \right) \left[\frac{U(w) - K - U(w_u)}{U'(w)} \right]$$

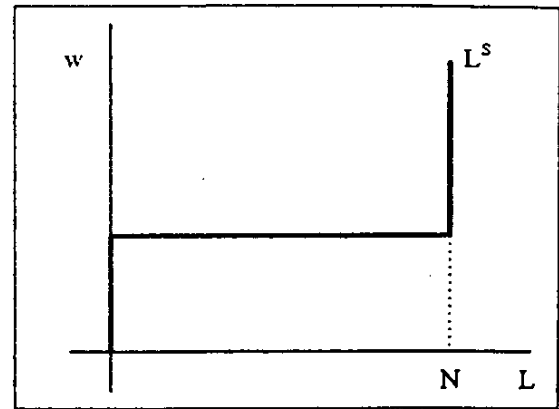
Although equation (6) only defines the union's choice of the wage implicitly, we can see that this choice does not depend on the shock to labor demand, A . From equation (3), since w does not vary with A , the elasticity of L with respect to A is:

$$(7) \frac{\partial L}{\partial A} \cdot \frac{A}{L} = \frac{\partial \ln L}{\partial \ln A} = \frac{1}{1-\alpha}$$

b) The union's objective function is:

$$(8) U^U = [U(w) - K]L + U(w_u) [N - L]$$

If the wage is such that $U(w) - K > U(w_u)$, then the union's objective function is maximized by having all of its members employed. That is, by choosing $L = N$. If the wage is such that $U(w) - K < U(w_u)$, the union's objective function is maximized by having no one employed. Finally, if the wage is such that $U(w) - K = U(w_u)$, the union is indifferent as to how many of its members are employed.



The union's labor supply curve under spot markets is depicted in the diagram at right. How the wage and employment vary with A will depend on whether the labor demand curve intersects the completely elastic or the completely inelastic portion of the labor supply curve.

So first of all, if the labor demand curve intersects the inelastic portion, L does not vary with A under spot markets — employment does not vary in response to shocks to labor demand. Rearranging equation (3) to solve for w yields:

$$(9) w = AL^{-1/(1-\alpha)} \quad \text{where } L = N$$

and thus the elasticity of the wage with respect to A is:

$$(10) \frac{\partial w}{\partial A} \cdot \frac{A}{w} = \frac{\partial \ln w}{\partial \ln A} = 1$$

However, if the labor demand curve intersects the elastic portion of the labor supply curve, we get the opposite result. The real wage does not respond at all to changes in A . From equation (3), this means that the elasticity of employment with respect to A is $1/(1 - \alpha)$.

c) i) With spot markets, the union takes w as given and chooses L to maximize $U^U = wL - [\sigma/(\sigma + 1)]L^{(\sigma+1)/\sigma}$. The first-order condition is:

$$(11) \partial U^U / \partial L = w - L^{1/\sigma} = 0$$

and thus the union's choice of labor supply as a function of the wage is:

$$(12) L = w^\sigma$$

and so σ represents the elasticity of labor supply with respect to the wage.

Under spot markets, w will adjust to equate labor demand and labor supply. Setting the right-hand sides of equations (3) and (12) equal to each other gives us:

$$(13) A^{1/(1-\alpha)} w^{-1/(1-\alpha)} = w^\sigma \Rightarrow Aw^{-1} = w^{\sigma(1-\alpha)}$$

and thus the spot market wage is:

$$(14) w = A^{1/[1+\sigma(1-\alpha)]}$$

The elasticity of the spot market wage with respect to the labor demand shocks is:

$$(15) \frac{\partial w}{\partial A} \cdot \frac{A}{w} = \frac{\partial \ln w}{\partial \ln A} = \frac{1}{1 + \sigma(1 - \alpha)}$$

Substituting equation (14) into either the demand or supply curve will give us equilibrium employment.

Substituting it into equation (12) yields:

$$(16) L = A^{\sigma/[1+\sigma(1-\alpha)]}$$

The elasticity of employment with respect to the labor demand shocks is:

$$(17) \frac{\partial L}{\partial A} \cdot \frac{A}{L} = \frac{\partial \ln L}{\partial \ln A} = \frac{\sigma}{1 + \sigma(1 - \alpha)}$$

c) ii) Substituting equation (3) for the firm's choice of L into the union's objective function yields:

$$(18) U^U = A^{1/(1-\alpha)} w^{-\alpha/(1-\alpha)} - [\sigma/(\sigma+1)] A^{(\sigma+1)/\sigma(1-\alpha)} w^{-(\sigma+1)/\sigma(1-\alpha)}$$

The union's problem is to choose w in order to maximize its utility as given in equation (18). The first-order condition is:

$$(19) \frac{\partial U^U}{\partial w} = \left(\frac{-\alpha}{1-\alpha} \right) A^{1/(1-\alpha)} w^{-1/(1-\alpha)} + \left(\frac{\sigma}{\sigma+1} \right) \left[\frac{\sigma+1}{\sigma(1-\alpha)} \right] A^{(\sigma+1)/\sigma(1-\alpha)} w^{[-(\sigma+1)-\sigma(1-\alpha)]/\sigma(1-\alpha)} = 0$$

Equation (19) can be written as:

$$(20) \alpha A^{1/(1-\alpha)} w^{-1/(1-\alpha)} = A^{(\sigma+1)/\sigma(1-\alpha)} w^{[-(\sigma+1)-\sigma(1-\alpha)]/\sigma(1-\alpha)}$$

Taking both sides of equation (20) to the exponent $\sigma(1-\alpha)$ yields:

$$(21) \alpha^{\sigma(1-\alpha)} A^{\sigma} w^{-\sigma} = A^{\sigma+1} w^{-(\sigma+1)-\sigma(1-\alpha)}$$

or simply:

$$(22) w^{1+\sigma(1-\alpha)} = \alpha^{\sigma(1-\alpha)} A^{-1}$$

and thus the union's choice of w is:

$$(23) w = \alpha^{\sigma(1-\alpha)/[1+\sigma(1-\alpha)]} A^{1/[1+\sigma(1-\alpha)]}$$

And thus the elasticity of this wage determined by the union with respect to the labor demand shocks is:

$$(24) \frac{\partial w}{\partial A} \cdot \frac{A}{w} = \frac{\partial \ln w}{\partial \ln A} = \frac{1}{1 + \sigma(1 - \alpha)}$$

This is exactly the same elasticity as when the wage was determined by the spot market. Thus with this union objective function, the behavior of the wage in response to labor demand shocks does not depend on whether or not unions get to set the wage or whether the wage is determined by supply and demand in a spot market.

Substituting equation (23) into the firm's choice of L given in equation (3) yields:

$$(25) L = A^{1/(1-\alpha)} \alpha^{-\alpha(1-\alpha)/(1-\alpha)[1+\sigma(1-\alpha)]} A^{-1/(1-\alpha)[1+\sigma(1-\alpha)]}$$

Combining the terms in A gives us:

$$L = \alpha^{-\sigma/[1+\sigma(1-\alpha)]} A^{[1+\sigma(1-\alpha)-1]/(1-\alpha)[1+\sigma(1-\alpha)]}$$

or simply:

$$(26) L = \alpha^{-\sigma/[1+\sigma(1-\alpha)]} A^{\sigma/[1+\sigma(1-\alpha)]}$$

The elasticity of equilibrium employment with respect to A is:

$$(27) \frac{\partial L}{\partial A} \cdot \frac{A}{L} = \frac{\partial \ln L}{\partial \ln A} = \frac{\sigma}{1 + \sigma(1 - \alpha)}$$

Again, this is exactly the same elasticity as in the spot market case. With this union objective function, whether there is a spot market or whether the union sets the wage, the behavior of employment in response to labor demand shocks is the same.

Problem 10.10

a) i) The union's problem is to choose w and L in order to maximize $[U(w) - K]L + U(w_s)[N - L]$ subject to $AL^\alpha/\alpha - wL \geq \pi_0$. Since the union has no reason to give the firm profits greater than π_0 , it satisfies the constraint with equality. The Lagrangian is:

$$(1) \mathcal{L} = [U(w) - K]L + U(w_s)[N - L] + \lambda[AL^\alpha/\alpha - wL - \pi_0]$$

a) ii) The first-order conditions are:

$$(2) \partial \mathcal{L} / \partial w = U'(w)L - \lambda L = 0 \quad \text{and} \quad (3) \partial \mathcal{L} / \partial L = U(w) - K - U(w_u) + \lambda AL^{\alpha-1} - \lambda w = 0$$

a) iii) The wage is the means by which the union extracts the rent from being able to reduce the firm's profits to its minimum required level.

a) iv) From equation (2), $\lambda = U'(w)$. Substituting this fact into equation (3) yields:

$$(4) U(w) - K - U(w_u) + U'(w)AL^{\alpha-1} - U'(w)w = 0$$

Rearranging equation (4) yields:

$$(5) U'(w)AL^{\alpha-1} = U'(w)w - U(w) + K + U(w_u)$$

and thus solving for the union's choice of L gives us:

$$(6) L = \left[\frac{U'(w)A}{U'(w)w - U(w) + K + U(w_u)} \right]^{1/(1-\alpha)}$$

From equation (6), the elasticity of employment with respect to the labor demand shocks is:

$$(7) \frac{\partial L}{\partial A} \cdot \frac{A}{L} = \frac{\partial \ln L}{\partial \ln A} = \frac{1}{1-\alpha} > 0$$

In part (b) of Problem 10.9, if labor demand intersected the elastic portion of labour supply, the elasticity of employment with respect to the labor demand shocks was the same as it is here.

b) Now the union's problem is to choose w and L in order to maximize $wL - [\sigma/(\sigma+1)]L^{(\sigma+1)/\sigma}$ subject to $AL^\alpha/\alpha - wL \geq \pi_0$. The Lagrangian is:

$$(8) \mathcal{L} = wL - [\sigma/(\sigma+1)]L^{(\sigma+1)/\sigma} + \lambda[AL^\alpha/\alpha - wL - \pi_0]$$

The first-order conditions are:

$$(9) \partial \mathcal{L} / \partial w = L - \lambda L = 0 \quad \text{and} \quad (10) \partial \mathcal{L} / \partial L = w - L^{1/\sigma} + \lambda AL^{\alpha-1} - \lambda w = 0$$

From equation (9), $\lambda = 1$. Substituting this fact into equation (10) gives us:

$$(11) w - L^{1/\sigma} + AL^{\alpha-1} - w = 0$$

Equation (11) can be written as:

$$L^{1/\sigma} = AL^{\alpha-1} \Rightarrow L^{[1-\sigma(\alpha-1)]/\sigma} = A$$

and thus the union's choice of L is

$$(12) L = A^{\sigma/[1+\sigma(1-\alpha)]}$$

The elasticity of employment with respect to the labor demand shocks is:

$$(13) \frac{\partial L}{\partial A} \cdot \frac{A}{L} = \frac{\partial \ln L}{\partial \ln A} = \frac{\sigma}{1+\sigma(1-\alpha)}$$

Note that this is exactly the same elasticity as in the spot market case – see equation (17) in Problem 10.9. Thus equilibrium employment varies in exactly the same way (due to the labor demand shocks) regardless of whether the union controls the wage and employment or whether the labor market equilibrium is determined by the forces of demand and supply.

Problem 10.11

a) In equilibrium, the number of people in the primary sector will be equal to the number of employed people – which in turn equals the number of primary sector jobs, N_p – plus the number of unemployed people in the economy, U . Since people are picked at random for jobs, in equilibrium, the probability of obtaining a primary sector job, q , is equal to the total number of jobs, N_p , divided by the equilibrium pool of primary sector workers, $N_p + U$. Thus in equilibrium:

$$(1) q = N_p / (N_p + U)$$

In addition, in equilibrium, the expected utility of choosing the primary sector — $qw_p + (1 - q)b$ — must equal the expected utility of choosing the secondary sector, w_s . Thus in equilibrium:

$$(2) \quad qw_p + (1 - q)b = w_s$$

Solving equation (2) for q gives us:

$$(3) \quad q = (w_s - b)/(w_p - b)$$

We have two conditions that q must satisfy in equilibrium. Setting the right-hand sides of equations (1) and (3) equal, we have:

$$N_p / (N_p + U) = (w_s - b)/(w_p - b) \Rightarrow N_p (w_p - b) = N_p (w_s - b) + (w_s - b)U$$

Solving for equilibrium unemployment we have:

$$(w_s - b)U = (w_p - w_s)N_p$$

or simply:

$$(4) \quad U = \left(\frac{w_p - w_s}{w_s - b} \right) N_p$$

b) To see how an increase in the number of primary sector jobs affects unemployment, take the derivative of U with respect to N_p :

$$(5) \quad \frac{\partial U}{\partial N_p} = \left(\frac{w_p - w_s}{w_s - b} \right) > 0$$

Note that w_s must be greater than b or no one would choose the secondary sector. Equation (5) says that a rise in the number of primary sector jobs actually increases equilibrium unemployment. The fact that there are more of these jobs does, for a given number of primary sector workers, increase the likelihood of getting a job. But that very fact encourages more people to choose the primary sector over the secondary sector. And indeed, so many more people choose the primary sector that the number of primary sector workers who do not get jobs actually rises.

c) To see the effects of an increase in the level of unemployment benefits, take the derivative of U with respect to b :

$$(6) \quad \frac{\partial U}{\partial b} = \frac{(w_p - w_s)}{(w_s - b)^2} N_p > 0$$

Thus unemployment rises when b rises. Again, the intuition is that higher unemployment benefits make the primary sector more attractive. Thus more people choose the primary sector. Since there are a fixed number of jobs in the primary sector, more people will end up unemployed.

Problem 10.12

a) $V = E(w - Cn') = E(w) - CE(n')$ and is the expected value of the wage the worker will eventually accept if she searches more minus the expected cost of further searching. The expected cost of further searching is the expected number of jobs to sampled multiplied by the (known) cost of sampling each job. Thus V can be interpreted as the expected value of further searching. Clearly, if the worker is offered a job that pays a wage of $\hat{w}$, where $\hat{w}$ exceeds the expected value of further searching, it is optimal to stop searching and take that job. However, if the wage offered is less than the expected value of further searching, it is optimal to reject that job and continue searching.

b) Note that:

$$(1) \quad V = F(V)V + \int_{w=V}^{\infty} wf(w)dw - C$$

can be rewritten as:

$$(2) \quad V = \frac{\int_{w=V}^{\infty} wf(w)dw}{1 - F(V)} - \frac{C}{1 - F(V)}$$

Consider the first term on the right-hand side of equation (2). The denominator is the probability that a wage drawn from the distribution will be greater than the cutoff, V . Thus this first term represents the expected wage conditional on that wage being greater than the reservation wage of V .

Now consider the second term on the right-hand side of equation (2). Since $1 - F(V)$ is the probability of drawing a wage greater than V , $1/[1 - F(V)]$ is the expected number of jobs that will need to be sampled in order to draw a job with a wage greater than V . For example if the probability of drawing a wage greater than V is $1/2$, then on average it will take two draws in order to obtain a wage greater than V . Therefore, $C/[1 - F(V)]$ is the expected cost of sampling jobs. Thus $V = E(w) - CE(n')$ must satisfy equation (2).

c) By Leibniz's rule and the chain rule, we have:

$$(3) \quad \frac{\partial \left[\int_{w=V(C)}^{\infty} wf(w)dw \right]}{\partial C} = \frac{\partial \left[\int_{w=V(C)}^{\infty} wf(w)dw \right]}{\partial V} \frac{\partial V}{\partial C} = -Vf(V) \frac{\partial V}{\partial C}$$

Differentiating both sides of equation (1) with respect to C and using the result in equation (3) yields:

$$(4) \quad \frac{\partial V}{\partial C} = \left[f(V) \frac{\partial V}{\partial C} \right] V + F(V) \frac{\partial V}{\partial C} - Vf(V) \frac{\partial V}{\partial C} - 1$$

Collecting terms in equation (4) gives us:

$$(5) \quad [1 - F(V)] \partial V / \partial C = -1$$

or simply:

$$(6) \quad \frac{\partial V}{\partial C} = \frac{-1}{1 - F(V)}$$

With $F(V) < 1$, $\partial V / \partial C > 0$ so that an increase in the cost of sampling jobs reduces the value of the reservation wage.

d) A searcher will never accept a job that she has previously rejected. From equation (2), V is a constant. If a searcher rejects a job paying some wage $\hat{w}$, it must mean that $\hat{w}$ is less than V and will always be less than V . Thus the worker will never accept the job paying $\hat{w}$.

Problem 10.13

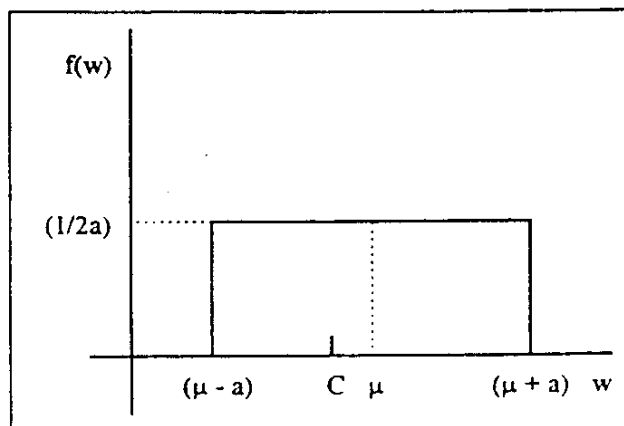
a) The distribution of wages is depicted at right. The cost of sampling a job, C , is also depicted under the assumption that $C < \mu$.

For this uniform distribution, the probability density function is:

$$(1) \quad f(w) = 1/2a$$

The associated cumulative distribution function is given by:

$$(2) \quad F(w) = \frac{w - (\mu - a)}{2a}$$



As explained in part (b) of Problem 10.12, V — the cutoff or reservation wage — must satisfy:

$$(3) \quad V = F(V)V + \int_{w=V}^{\mu+a} wf(w)dw - C$$

Note that we can write the upper bound of the integral as $(\mu + a)$ since $f(w) = 0$ for all $w > (\mu + a)$.

Substituting equations (1) and (2) into equation (3) yields:

$$(4) \quad V = \left[\frac{V - (\mu - a)}{2a} \right] V + \int_{w=V}^{\mu+a} (w/2a) dw - C$$

The value of the integral in equation (4) is:

$$(5) \quad \int_{w=V}^{\mu+a} (w/2a) dw = \frac{1}{2a} \left[\frac{1}{2} w^2 \right]_{w=V}^{w=\mu+a} = \frac{1}{4a} [(\mu + a)^2 - V^2]$$

Substituting equation (5) into equation (4) and multiplying both sides of the resulting expression by $4a$ gives us:

$$(6) \quad 4aV = 2V^2 - 2(\mu - a)V + (\mu + a)^2 - V^2 - 4aC$$

Collecting terms yields:

$$(7) \quad V^2 - 2(\mu + a)V + (\mu + a)^2 - 4aC = 0$$

Using the quadratic formula, we have:

$$(8) \quad V = \frac{2(\mu + a) \pm \sqrt{4(\mu + a)^2 - 4(\mu + a)^2 + 16aC}}{2} = \frac{2(\mu + a) \pm 4\sqrt{aC}}{2}$$

We can ignore the solution with $V > (\mu + a)$ since $(\mu + a)$ is the highest possible wage. Thus V is given by:

$$(9) \quad V = (\mu + a) - 2a^{1/2} C^{1/2}$$

Note that if $C = 0$ — there is no cost to sampling a job — $V = (\mu + a)$ meaning that the worker simply keeps searching until she is offered the highest wage in the distribution. In addition, if $C = a$, $V = (\mu + a) - 2a$ or $V = (\mu - a)$ meaning that the worker will accept any wage. Finally, if $C > a$, $V < (\mu + a)$ and so again, the worker accepts any wage that is offered.

To see how V varies with a (which measures the dispersion of wages), use equation (9) to find the derivative of V with respect to a :

$$(10) \quad \partial V / \partial a = 1 - a^{-1/2} C^{1/2} = 1 - (C/a)^{1/2}$$

With $C < a$, a rise in a increases the reservation wage, V . The fact that there are now more higher paying jobs increases the value of further searching and thus increases the cutoff wage.

Problem 10.14

a) Dividing both sides of equation (10.82) in the text by r gives us the value of a vacancy:

$$(1) \quad V_v = -\frac{C}{r} + \frac{\alpha A}{r(a + \alpha + 2b + 2r)}$$

From equation (1), an increase in b directly reduces V_v for a given level of employment. However, α and a also depend on b and thus we must examine how a rise in b affects them for a given level of employment.

From equation (10.83) in the text, a — the rate at which unemployed workers find jobs — is:

$$(2) \quad a = \frac{bE}{L - E}$$

Thus a rise in b increases a for a given level of employment. From equation (1), a rise in a reduces V_v for a given level of employment.

From equation (10.84) in the text, α — the rate at which vacancies are filled — is:

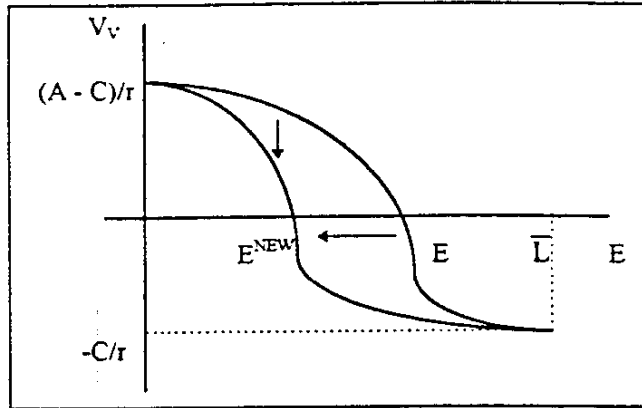
$$(3) \alpha = K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma}$$

With $\gamma < 1$, a rise in b reduces α for a given level of employment. From equation (1):

$$(4) \frac{\partial V_V}{\partial \alpha} = \frac{A[r(a + \alpha + 2b + 2r)] - \alpha Ar}{[r(a + \alpha + 2b + 2r)]^2} = \frac{A[r(a + 2b + 2r)]}{[r(a + \alpha + 2b + 2r)]^2} > 0$$

Thus a fall in α reduces V_V for a given level of employment.

In summary, all of these effects work in the same direction. The rise in the job breakup rate, b , reduces V_V for a given level of employment. Thus the V_V locus shifts down (except as $E \rightarrow \bar{L}$ and as $E \rightarrow 0$). See the diagram at right. The equilibrium level of employment – which is given by the intersection of the V_V locus with the free-entry condition that $V_V = 0$ – falls from E to E^{NEW} .



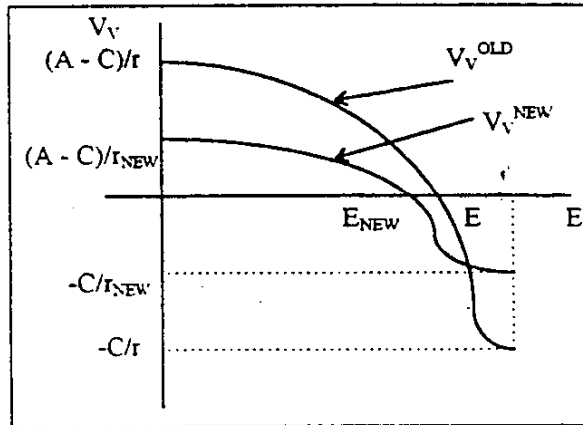
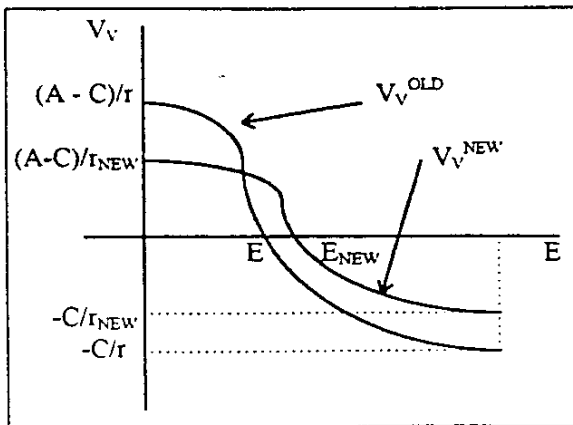
b) The increase in r means that as $E \rightarrow 0$, the value of a vacancy is now $(A - C)/r_{\text{NEW}}$ which is less than $(A - C)/r$. As $E \rightarrow \bar{L}$, the value of a vacancy is $-C/r_{\text{NEW}}$ which is greater than $-C/r$. Thus the V_V locus must shift down at lower levels of E and shift up at higher levels of E . More formally, at a given level of employment, since a and α do not depend upon r , we have:

$$(5) \frac{\partial V_V}{\partial r} = \frac{C}{r^2} - \frac{\alpha A(a + \alpha + 2b + 4r)}{[r(a + \alpha + 2b + 2r)]^2}$$

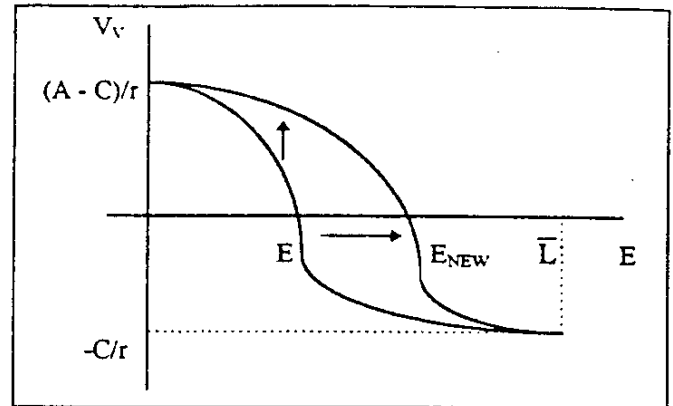
The level of E at which the new V_V locus intersects the old V_V locus is defined by $\partial V_V / \partial r = 0$ or:

$$\frac{C}{r^2} = \frac{\alpha A(a + \alpha + 2b + 4r)}{r^2(a + \alpha + 2b + 2r)^2} \Rightarrow C(a + \alpha + 2b + 2r)^2 = \alpha A(a + \alpha + 2b + 4r) \quad (6)$$

Substituting $a = bE/(\bar{L} - E)$ and $\alpha = K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma}$ into equation (6) would yield an equation that implicitly defined the level of E at which the old and new V_V loci intersect. If the original level of employment exceeds the intersection level of E , equilibrium employment rises. This is the case depicted in the diagram on the left. If the original level of employment is less than the intersection level of E , equilibrium employment falls. This is the case depicted in the diagram on the right.



c) At a given level of employment, $a = bE/(\bar{L} - E)$ does not depend upon K . At a given level of employment, $\alpha = K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma}$, is increasing in K . As shown in equation (4), a rise in α causes V_V to rise for a given level of employment. Thus an increase in the effectiveness of matching shifts the V_V locus up (except as $E \rightarrow \bar{L}$ and as $E \rightarrow 0$). As depicted in the diagram at right, the equilibrium level of employment rises.



Problem 10.15

a) Substituting equation (10.77) — $(V_E - V_U) = w/(a + b + r)$ — and equation (10.78) — $(V_F - V_V) = (A - w)/(\alpha + b + r)$ — into the assumption that fraction f of the surplus goes to the worker while fraction $(1 - f)$ goes to the firm yields:

$$(1) \quad (1 - f) \left[\frac{w}{a + b + r} \right] = f \left[\frac{A - w}{\alpha + b + r} \right]$$

We need to solve for w . Rearranging equation (1) and obtaining a common denominator gives us:

$$(2) \quad \frac{w(\alpha + b + r) - w[f(\alpha + b + r)] + w[f(a + b + r)]}{(a + b + r)(\alpha + b + r)} = \frac{fA}{\alpha + b + r}$$

or simply:

$$(3) \quad w[\alpha(1 - f) + fa + b + r] = fA(a + b + r)$$

and thus w is given by:

$$(4) \quad w = \frac{fA(a + b + r)}{fa + (1 - f)\alpha + b + r}$$

Substituting equation (4) into equation (10.77) — $(V_E - V_U) = w/(a + b + r)$ — yields:

$$(5) \quad V_E - V_U = \frac{fA(a + b + r)}{[fa + (1 - f)\alpha + b + r](a + b + r)} = \frac{fA}{fa + (1 - f)\alpha + b + r}$$

Equation (10.75) in the text states that rV_V equals $-C + \alpha(V_F - V_V)$. Our assumption about how the surplus is split implies that $V_F - V_V = [(1 - f)/f][V_E - V_U]$. Thus:

$$(6) \quad rV_V = -C + \alpha[(1 - f)/f][V_E - V_U]$$

Substituting equation (5) into equation (6) yields:

$$(7) \quad rV_V = -C + \frac{\alpha(1 - f)}{f} \cdot \frac{fA}{fa + (1 - f)\alpha + b + r} = -C + \frac{(1 - f)\alpha A}{fa + (1 - f)\alpha + b + r}$$

It is straightforward to verify that equation (7) reduces to equation (10.82) in the text when $f = 1/2$.

Substituting equation (10.83) — $a = bE/(\bar{L} - E)$ — and equation (10.84) — $\alpha = K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma}$ — and imposing $V_V = 0$, equation (7) becomes:

$$(8) \quad \frac{(1 - f)K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma}}{[fbE/(\bar{L} - E)] + (1 - f)K^{1/\gamma} (bE)^{(\gamma-1)/\gamma} (\bar{L} - E)^{\beta/\gamma} + b + r} A = C$$

Equation (8) is analogous to equation (10.85) in the text.

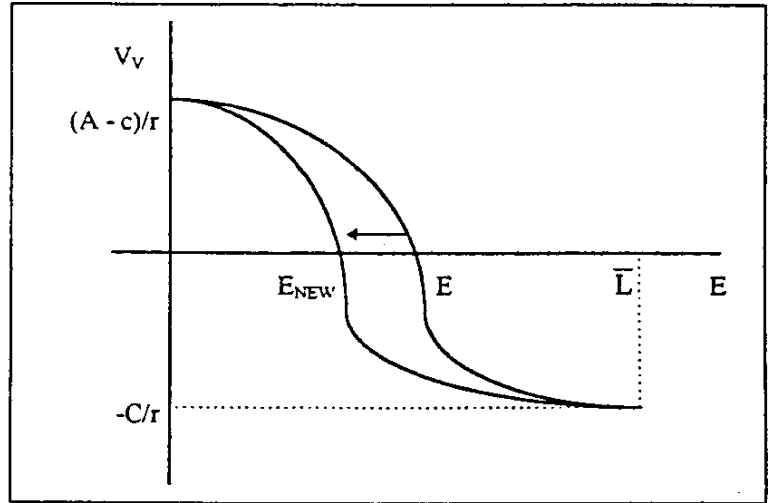
b) For a given level of E , a and α do not depend on f . Thus we can use equation (7) to examine the effects of a change in f on the V_V locus. We have, for a given level of employment:

$$(9) \quad \frac{\partial r V_V}{\partial f} = \frac{-\alpha A [f\alpha + (1-f)\alpha + b + r] - (1-f)\alpha A (a - \alpha)}{f\alpha + (1-f)\alpha + b + r}$$

The sign of $\partial r V_V / \partial f$ will be determined by the sign of the numerator in equation (9). Simplifying that numerator, we have:

$$\begin{aligned} & -\alpha A [f\alpha + (1-f)\alpha + b + r] - (1-f)\alpha A (a - \alpha) = \\ & -f\alpha A a - (1-f)\alpha^2 A - \alpha A - \alpha A (b + r) - \alpha A a + f\alpha A a + (1-f)\alpha^2 A = -\alpha A (a + b + r) < 0 \end{aligned}$$

Thus an increase in f causes V_V to be lower for a given level of E . That is, it causes the V_V locus to shift down. As the diagram depicts, an increase in the fraction of the surplus that goes to the worker reduces the equilibrium level of employment. Intuitively, the only decision that a firm has is whether to enter or not. If f rises, so that the worker gets a bigger fraction of the surplus from a job, entry is less attractive. Thus the equilibrium level of employment falls.



Problem 10.16

After the fall in A , there is no reason for firms whose positions are filled to discharge their workers. Thus employment and unemployment do not change discontinuously at the time of the shock. The reduced attractiveness of hiring does cause the value of a vacancy, V_V , to fall. However, since exiting is not allowed, we do not require $V_V = 0$ and so vacancies do not change. Since employment, unemployment and vacancies are not affected at the time of the fall in A , the number of new matches, $M = KU^\beta V^\gamma$, continues to equal the flows into unemployment, bE . In summary, if we rule out entry and exit, unemployment and vacancies do not respond at all to the fall in A .

Problem 10.17

a) In a steady state, $M(U, V) = bE(N)$. The number of matches per unit time must equal the number of jobs that end per unit time. In addition, the number of unemployed workers is $U = \bar{L} - E(N)$ and the number of vacancies is $V = N - E(N)$. Substituting these expressions for U and V into the matching function — $M(U, V) = KU^\beta V^\gamma$ — and setting it equal to $bE(N)$, we have:

$$(1) \quad bE(N) = K[\bar{L} - E(N)]^\beta [N - E(N)]^\gamma$$

To find how E varies with N , differentiate both sides of equation (1) with respect to N :

$$(2) \quad bE'(N) = K\beta[\bar{L} - E(N)]^{\beta-1} [N - E(N)]^\gamma [-E'(N)] + K[\bar{L} - E(N)]^\beta \gamma [N - E(N)]^{\gamma-1} [1 - E'(N)]$$

Simplifying yields:

$$(3) \quad bE'(N) = \frac{K\beta[\bar{L} - E(N)]^\beta [N - E(N)]^\gamma [-E'(N)]}{\bar{L} - E(N)} + \frac{K\gamma[\bar{L} - E(N)]^\beta [N - E(N)]^{\gamma-1} [1 - E'(N)]}{N - E(N)}$$

Using equation (1) for $bE(N)$, equation (3) can be written as:

$$(4) \quad bE'(N) = \frac{\beta bE(N)}{\bar{L} - E(N)} [-E'(N)] + \frac{\gamma bE(N)}{N - E(N)} [1 - E'(N)]$$

Collecting the terms in $E'(N)$ and dividing through by b yields:

$$(5) E'(N) \left[1 + \frac{\beta E(N)}{\bar{L} - E(N)} + \frac{\gamma E(N)}{N - E(N)} \right] = \frac{\gamma E(N)}{N - E(N)}$$

or simply:

$$(6) E'(N) = \frac{\frac{\gamma E(N)}{N - E(N)}}{1 + \frac{\beta E(N)}{\bar{L} - E(N)} + \frac{\gamma E(N)}{N - E(N)}} = \frac{\gamma E(N) [\bar{L} - E(N)]}{[N - E(N)] [\bar{L} - E(N)] + \beta E(N) [N - E(N)] + \gamma E(N) [\bar{L} - E(N)]}$$

b) Since welfare is given by $W(N) = AE(N) - NC$, the change in welfare due to a change in the number of jobs is given by:

$$(7) W'(N) = AE'(N) - C$$

Substituting equation (6) into equation (7) yields:

$$(8) E'(N) = \frac{\gamma E(N) [\bar{L} - E(N)]}{[N - E(N)] [\bar{L} - E(N)] + \beta E(N) [N - E(N)] + \gamma E(N) [\bar{L} - E(N)]} A - C$$

c) To simplify the notation, we can drop the "EQ" subscripts on N – keep in mind that N is the number of jobs in equilibrium. In the case of $r = 0$, equation (10.82) in the text simplifies to:

$$(9) C = \alpha A / (a + \alpha + 2b)$$

Substituting $a = bE(N)/[\bar{L} - E(N)]$ and $\alpha = bE(N)/V(N)$ into equation (9) gives us:

$$(10) C = \frac{bE(N)/V(N)}{bE(N)/[\bar{L} - E(N)] + bE(N)/V(N) + 2b} A$$

Multiplying the top and bottom of the right-hand side of equation (10) by $V(N)[\bar{L} - E(N)]/b$ yields:

$$(11) C = \frac{E(N) [\bar{L} - E(N)]}{E(N)V(N) + E(N) [\bar{L} - E(N)] + 2V(N) [\bar{L} - E(N)]} A$$

Finally, using the definition of $V(N) = N - E(N)$, equation (11) can be written as:

$$(12) C = \frac{E(N) [\bar{L} - E(N)]}{E(N) [N - E(N)] + E(N) [\bar{L} - E(N)] + 2[N - E(N)] [\bar{L} - E(N)]} A$$

d) Using the definitions of $U(N) = \bar{L} - E(N)$ and $V(N) = N - E(N)$, equation (12) for C can be rewritten as:

$$(13) C = \frac{E(N)U(N)}{E(N)V(N) + E(N)U(N) + 2V(N)U(N)} A$$

Substituting equation (13) into equation (8) gives us an expression for how welfare changes with equilibrium employment:

$$(14) W'(N) = A \left\{ \frac{\gamma E(N)U(N)}{V(N)U(N) + \beta E(N)V(N) + \gamma E(N)U(N)} - \frac{E(N)U(N)}{E(N)V(N) + E(N)U(N) + 2V(N)U(N)} \right\}$$

After obtaining a common denominator, the sign of $W'(N)$ will be determined by the sign of:

$$\gamma U(N)[E(N)V(N) + E(N)U(N) + 2V(N)U(N)] - U(N)[V(N)U(N) + \beta E(N)V(N) + \gamma E(N)U(N)]$$

which simplifies to:

$$(\gamma - \beta)U(N)E(N)V(N) + (\gamma - \gamma)E(N)U(N)^2 + (2\gamma - 1)V(N)U(N)^2 = U(N)V(N)[(\gamma - \beta)E(N) + (2\gamma - 1)U(N)]$$

Thus the sign of $W'(N)$ will be determined by the sign of $(\gamma - \beta)E(N) + (2\gamma - 1)U(N)$. Finally, using the fact that $U(N) = \bar{L} - E(N)$, the sign of $W'(N)$ will be determined by:

$$(15) \text{sign}[(\gamma - \beta)E(N) + (2\gamma - 1)\bar{L} - (2\gamma - 1)E(N)] = \text{sign}[(2\gamma - 1)\bar{L} + (1 - \gamma - \beta)E(N)]$$

If $\beta + \gamma = 1$ -- if matching has constant returns -- the sign of $W'(N)$ is determined by the sign of $(2\gamma - 1)$. If $\gamma > 1/2$, $W'(N) > 0$ and thus an increase in the equilibrium number of jobs would raise welfare or in other words, equilibrium unemployment is inefficiently high. But if $\gamma < 1/2$, $W'(N) < 0$ and thus an increase in N reduces welfare and so equilibrium unemployment is inefficiently low.

If $\gamma = 1/2$, the sign of $W'(N)$ is determined by the sign of $(1 - \gamma - \beta) = [(1/2) - \beta]$. Thus if $\beta < 1/2$ so that $\gamma + \beta < 1$ -- matching has decreasing returns -- $W'(N) > 0$ and so equilibrium unemployment is inefficiently high. But if $\beta > 1/2$ so that $\gamma + \beta > 1$ -- matching has increasing returns -- $W'(N) < 0$ and so equilibrium unemployment is inefficiently low. Thus a greater role of unemployment in creating matches (that is, a larger value of β given γ) makes it more likely that the decentralized equilibrium involves too many jobs.

Problems

- 1.1. Consider a Solow economy that is on its balanced growth path. Assume for simplicity that there is no technological progress. Now suppose that the rate of population growth falls.
 - (a) What happens to the balanced-growth-path values of capital per worker, output per worker, and consumption per worker? Sketch the paths of these variables as the economy moves to its new balanced growth path.
 - (b) Describe the effect of the fall in population growth on the path of output (that is, total output, not output per worker).
- 1.2. Suppose that the production function is Cobb-Douglas.
 - (a) Find expressions for k^* , y^* , and c^* as functions of the parameters of the model, s , n , δ , g , and α .
 - (b) What is the golden-rule value of k ?
 - (c) What saving rate is needed to yield the golden-rule capital stock?
- 1.3. Consider the constant elasticity of substitution (CES) production function, $Y = [K^{(\sigma-1)/\sigma} + (AL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)}$, where $0 < \sigma < \infty$ and $\sigma \neq 1$. (σ is the elasticity of substitution between capital and effective labor. In the special case of $\sigma = 1$, the CES function reduces to the Cobb-Douglas.)
 - (a) Show that this production function exhibits constant returns to scale.
 - (b) Find the intensive form of the production function.
 - (c) Under what conditions does the intensive form satisfy $f'(\bullet) > 0$, $f''(\bullet) < 0$?
 - (d) Under what conditions does the intensive form satisfy the Inada conditions?
- 1.4. Consider an economy with technological progress but without population growth that is on its balanced growth path. Now suppose there is a one-time jump in the number of workers.
 - (a) At the time of the jump, does output per unit of effective labor rise, fall, or stay the same? Why?
 - (b) After the initial change (if any) in output per unit of effective labor when the new workers appear, is there any further change in output per unit of effective labor? If so, does it rise or fall? Why?
 - (c) Once the economy has again reached a balanced growth path, is output per unit of effective labor higher, lower, or the same as it was before the new workers appeared? Why?
- 1.5. Find the elasticity of output per unit of effective labor on the balanced growth path, y^* , with respect to the rate of population growth, n . If $\alpha_K(k^*) = \frac{1}{3}$, $g = 2\%$, and $\delta = 3\%$, by about how much does a fall in n from 2% to 1% raise y^* ?
- 1.6. Suppose that, despite the political obstacles, the United States permanently reduces its budget deficit from 3% of GDP to zero. Suppose that initially $s = 0.15$ and that investment rises by the full amount of the fall in the deficit. Assume that capital's share is $\frac{1}{3}$.

- (a) By about how much does output eventually rise relative to what it would have been without the deficit reduction?
- (b) By about how much does consumption rise relative to what it would have been without the deficit reduction?
- (c) What is the immediate effect of the deficit reduction on consumption? About how long does it take for consumption to return to what it would have been without the deficit reduction?

1.7. Factor payments in the Solow model. Assume that both labor and capital are paid their marginal products. Let w denote $\partial F(K, AL)/\partial L$ and r denote $\partial F(K, AL)/\partial K$.

- (a) Show that the marginal product of labor, w , is $A[f(k) - kf'(k)]$.
- (b) Show that if both capital and labor are paid their marginal products, constant returns to scale implies that the total amount paid to the factors of production equals total output. That is, show that under constant returns, $wL + rK = F(K, AL)$.
- (c) Two additional stylized facts about growth listed by Kaldor (1961) are that the return to capital (r) is approximately constant and that the shares of output going to capital and labor are each roughly constant. Does a Solow economy on a balanced growth path exhibit these properties? What are the growth rates of w and r on a balanced growth path?
- (d) Suppose the economy begins with a level of k less than k^* . As k moves toward k^* , is w growing at a rate greater than, less than, or equal to its growth rate on the balanced growth path? What about r ?

1.8. Suppose that, as in Problem 1.7, capital and labor are paid their marginal products. In addition, suppose that all capital income is saved and all labor income is consumed. Thus $\dot{K} = [\partial F(K, AL)/\partial K]K - \delta K$.

- (a) Show that this economy converges to a balanced growth path.
- (b) Is k on the balanced growth path greater than, less than, or equal to the golden-rule level of k ? What is the intuition for this result?

1.9. The Harrod-Domar model. (See Harrod, 1939, and Domar, 1946.) Suppose the production function is Leontief, $Y(t) = \min[c_K K(t), c_L e^{gt} L(t)]$, where c_K , c_L , and g are all positive. As in the Solow model, $\dot{L}(t) = nL(t)$ and $\dot{K}(t) = sY(t) - \delta K(t)$. Finally, assume $c_K K(0) = c_L L(0)$.

- (a) Under what condition does $c_K K(t) = c_L e^{gt} L(t)$ for all t ? If c_K , c_L , g , s , δ , and n are determined by separate considerations, is there any reason to expect that this condition holds?
- (b) If $c_L e^{gt} L(t)$ is growing faster than $c_K K(t)$ (and if the excess labor is assumed to be unemployed), what happens to the unemployment rate over time?
- (c) If $c_K K(t)$ is growing faster than $c_L e^{gt} L(t)$ (and if the excess capital is assumed to be unused), what happens to the fraction of the capital stock that is used over time?

1.10. Natural resources in the Solow model. At least since Malthus, some have argued that the fact that some factors of production (notably land and natural

resources) are available in finite supply must eventually bring growth to a halt. This problem asks you to address this idea in the context of the Solow model.

Let the production function be $Y = K^\alpha (AL)^\beta R^{1-\alpha-\beta}$, where R is the amount of land. Assume $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$. The factors of production evolve according to $\dot{K} = sY - \delta K$, $\dot{A} = gA$, $\dot{L} = nL$, and $\dot{R} = 0$.

- (a) Does this economy have a unique and stable balanced growth path? That is, does the economy converge to a situation in which each of Y , K , L , A , and R are growing at constant (but not necessarily equal) rates? If so, what are those growth rates? If not, why not?
- (b) In light of your answer, does the fact that the stock of land is constant imply that permanent growth is not possible? Explain intuitively.

1.11. Embodied technological progress. (This follows Solow, 1960, and Sato, 1966.) One view of technological progress is that the productivity of capital goods built at t depends on the state of technology at t and is unaffected by subsequent technological progress. This is known as *embodied technological progress* (technological progress must be “embodied” in new capital before it can raise output). This problem asks you to investigate its effects.

- (a) As a preliminary, let us modify the basic Solow model to make technological progress capital-augmenting rather than labor-augmenting. So that a balanced growth path exists, assume that the production function is Cobb-Douglas: $Y(t) = [A(t)K(t)]^\alpha L(t)^{1-\alpha}$. Assume that A grows at rate μ : $\dot{A}(t) = \mu A(t)$.

Show that the economy converges to a balanced growth path, and find the growth rates of Y and K on the balanced growth path. (Hint: show that we can write $Y/(A^\phi L)$ as a function of $K/(A^\phi L)$, where $\phi = \alpha/(1-\alpha)$. Then analyze the dynamics of $K/(A^\phi L)$.)

- (b) Now consider embodied technological progress. Specifically, let the production function be $Y(t) = J(t)^\alpha L(t)^{1-\alpha}$, where $J(t)$ is the effective capital stock. The dynamics of $J(t)$ are given by $\dot{J}(t) = sA(t)Y(t) - \delta J(t)$. The presence of the $A(t)$ term in this expression means that the productivity of investment at t depends on the technology at t .

Show that the economy converges to a balanced growth path. What are the growth rates of Y and J on the balanced growth path? (Hint: let $\bar{J}(t) = J(t)/A(t)$. Then use the same approach as in (a), focusing on $\bar{J}/(A^\phi L)$ instead of $K/(A^\phi L)$.)

- (c) What is the elasticity of output on the balanced growth path with respect to s ?
- (d) In the vicinity of the balanced growth path, how rapidly does the economy converge to the balanced growth path?
- (e) Compare your results for (c) and (d) with the corresponding results in the text for the basic Solow model.

1.12. Consider a Solow economy on its balanced growth path. Suppose the growth-accounting techniques described in Section 1.7 are applied to this economy.

- (a) What fraction of growth in output per worker does growth accounting attribute to growth in capital per worker? What fraction does it attribute to technological progress?
 - (b) How can you reconcile your results in (a) with the fact that the Solow model implies that the growth rate of output per worker on the balanced growth path is determined solely by the rate of technological progress?
- 1.13. (a) In the model of convergence and measurement error in equations (1.33)–(1.34), suppose the true value of b is -1 . Does a regression of $\ln(Y/N)_{1979} - \ln(Y/N)_{1870}$ on a constant and $\ln(Y/N)_{1870}$ yield a biased estimate of b ? Explain.
- (b) Suppose there is measurement error in measured 1979 income per capita but not in 1870 income per capita. Does a regression of $\ln(Y/N)_{1979} - \ln(Y/N)_{1870}$ on a constant and $\ln(Y/N)_{1870}$ yield a biased estimate of b ? Explain.

of bonds, the government can thus cause the balanced-growth-path value of k to equal its golden-rule value.³⁶

Problems

2.1. Consider N firms each with the constant returns to scale production function $Y = F(K, AL)$, or (using the intensive form) $Y = ALf(k)$. Assume $f'(\bullet) > 0$, $f''(\bullet) < 0$. Assume that all firms can hire labor at wage wA and rent capital at cost r , and that all firms have the same value of A .

(a) Consider the problem of a firm trying to produce Y units of output at minimum cost. Show that the cost-minimizing level of k is uniquely defined and is independent of Y , and that all firms therefore choose the same value of k .

(b) Show that the total output of the N cost-minimizing firms equals the output that a single firm with the same production function has if it uses all of the labor and capital used by the N firms.

2.2. The elasticity of substitution with constant-relative-risk-aversion utility.

Consider an individual who lives for two periods and whose utility is given by equation (2.46). Let P_1 and P_2 denote the prices of consumption in the two periods, and let W denote the value of the individual's lifetime income; thus the budget constraint is $P_1 C_1 + P_2 C_2 = W$.

(a) What are the individual's utility-maximizing choices of C_1 and C_2 given P_1 , P_2 , and W ?

(b) The elasticity of substitution between consumption in the two periods is $-[(P_1/P_2)/(C_1/C_2)][\partial(C_1/C_2)/\partial(P_1/P_2)]$, or $-\partial \ln(C_1/C_2)/\partial \ln(P_1/P_2)$. Show that with the utility function (2.46), the elasticity of substitution between C_1 and C_2 is $1/\theta$.

2.3. Assume that the instantaneous utility function $u(C)$ in equation (2.1) is $\ln C$. Consider the problem of a household maximizing (2.1) subject to (2.5). Find an expression for C at each time as a function of initial wealth plus the present value of labor income, the path of $r(t)$, and the parameters of the utility function.

³⁶This policy involves no costs to the government, and no taxes, on the balanced growth path. When $k = k_{GR}$, $1 + f'(k) = (1 + n)(1 + g)$ (see Problem 2.14). Thus the amount the government needs, per unit of effective labor, to pay off its existing debt in period t is $(1 + n)(1 + g)b_t$. But the amount of new debt it issues in t is b_{t+1} per unit of period- $t + 1$ effective labor; this is $(1 + n)(1 + g)b_{t+1}$ per unit of period- t effective labor. When b is constant, these two quantities are equal, and so the new debt issues are just enough to pay off the outstanding debt.

Finally, if the economy begins with $k > k_{GR}$, the government needs to levy taxes to move k to its golden-rule level. Specifically, suppose $k_0 > k_{GR}$. If the government levies lump-sum taxes per unit of effective labor of amount $(1 - \alpha)k_0^\alpha - (1 - \alpha)k_{GR}^\alpha$ in period 0, the saving of the young per unit of period-1 effective labor is $[1/(1 + n)(1 + g)][1/(2 + \rho)](1 - \alpha)k_{GR}^\alpha = a_{GR}$. With $b_1 = a_{GR} - k_{GR}$, this ensures that $k_1 = k_{GR}$. The government can use the revenue from the taxes and the initial bond issue to increase the consumption of the period-0 old.

- 2.4. Consider a household with utility given by (2.1)–(2.2). Assume that the real interest rate is constant, and let W denote the household's initial wealth plus the present value of its lifetime labor income (the right-hand side of [2.5]). Find the utility-maximizing path of C given r , W , and the parameters of the utility function.
- 2.5. **The productivity slowdown and saving.** Consider a Ramsey–Cass–Koopmans economy that is on its balanced growth path, and suppose there is a permanent fall in g .
- How, if at all, does this affect the $\dot{k} = 0$ curve?
 - How, if at all, does this affect the $\dot{c} = 0$ curve?
 - What happens to c at the time of the change?
 - Find an expression for the impact of a marginal change in g on the fraction of output that is saved on the balanced growth path. Can one tell whether this expression is positive or negative?
 - For the case where the production function is Cobb–Douglas, $f(k) = k^\alpha$, rewrite your answer to part (d) in terms of ρ , n , g , θ , and α . (Hint: use the fact that $f'(k^*) = \rho + \theta g$.)
- 2.6. Describe how each of the following affect the $\dot{c} = 0$ and $\dot{k} = 0$ curves in Figure 2.5, and thus how they affect the balanced-growth-path values of c and k :
- A rise in θ .
 - A downward shift of the production function.
 - A change in the rate of depreciation from the value of zero assumed in the text to some positive level.
- 2.7. Derive an expression analogous to (2.37) for the case of a positive depreciation rate.
- 2.8. **Capital taxation in the Ramsey–Cass–Koopmans model.** Consider a Ramsey–Cass–Koopmans economy that is on its balanced growth path. Suppose that at some time, which we will call time 0, the government switches to a policy of taxing investment income at rate τ . Thus the real interest rate that households face is now given by $r(t) = (1 - \tau)f'(k(t))$. Assume that the government returns the revenue it collects from this tax through lump-sum transfers. Finally, assume that this change in tax policy is unanticipated.
- How does the tax affect the $\dot{c} = 0$ locus? The $\dot{k} = 0$ locus?
 - How does the economy respond to the adoption of the tax at time 0? What are the dynamics after time 0?
 - How do the values of c and k on the new balanced growth path compare with their values on the old balanced growth path?
 - (This is based on Barro, Mankiw, and Sala-i-Martin, 1995.) Suppose there are many economies like this one. Workers' preferences are the same in each country, but the tax rates on investment income may vary across countries. Assume that each country is on its balanced growth path.

- (i) Show that the saving rate on the balanced growth path, $(y^* - c^*)/y^*$, is decreasing in τ .
- (ii) Do citizens in low- τ , high- k^* , high-saving countries have any incentive to invest in low-saving countries? Why or why not?
- (e) Does your answer to part (c) imply that a policy of *subsidizing* investment (that is, making $\tau < 0$), and raising the revenue for this subsidy through lump-sum taxes, increases welfare? Why or why not?
- (f) How, if at all, do the answers to parts (a) and (b) change if the government does not rebate the revenue from the tax but instead uses it to make government purchases?

2.9. Using the phase diagram to analyze the impact of an anticipated change. Consider the policy described in Problem 2.8, but suppose that instead of announcing and implementing the tax at time 0, the government announces at time 0 that at some later time, time t_1 , investment income will begin to be taxed at rate τ .

- (a) Draw the phase diagram showing the dynamics of c and k after time t_1 .
- (b) Can c change discontinuously at time t_1 ? Why or why not?
- (c) Draw the phase diagram showing the dynamics of c and k before t_1 .
- (d) In light of your answers to parts (a), (b), and (c), what must c do at time 0?
- (e) Summarize your results by sketching the paths of c and k as functions of time.

2.10. Using the phase diagram to analyze the impact of unanticipated and anticipated temporary changes. Analyze the following two variations on Problem 2.9:

- (a) At time 0, the government announces that it will tax investment income at rate τ from time 0 until some later date t_1 ; thereafter investment income will again be untaxed.
- (b) At time 0, the government announces that from time t_1 to some later time t_2 , it will tax investment income at rate τ ; before t_1 and after t_2 , investment income will not be taxed.

2.11. The analysis of government policies in the Ramsey-Cass-Koopmans model in the text assumes that government purchases do not affect utility from private consumption. The opposite extreme is that government purchases and private consumption are perfect substitutes. Specifically, suppose that the utility function (2.14) is modified to be

$$U = B \int_{t=0}^{\infty} e^{-\beta t} \frac{[c(t) + G(t)]^{1-\theta}}{1-\theta} dt.$$

If the economy is initially on its balanced growth path and if households' preferences are given by U , what are the effects of a temporary increase in government purchases on the paths of consumption, capital, and the interest rate?

2.12. Precautionary saving, non-lump-sum taxation, and Ricardian equivalence.

(This follows Leland, 1968, and Barsky, Mankiw, and Zeldes, 1986.) Consider an individual who lives for two periods. The individual has no initial wealth and earns labor incomes of amounts Y_1 and Y_2 in the two periods. Y_1 is known, but Y_2 is random; assume for simplicity that $E[Y_2] = Y_1$. The government taxes income at rate τ_1 in period 1 and τ_2 in period 2. The individual can borrow and lend at a fixed interest rate, which for simplicity is assumed to be zero. Thus second-period consumption is $C_2 = [(1 - \tau_1)Y_1 - C_1] + (1 - \tau_2)Y_2$. The individual chooses C_1 to maximize expected lifetime utility, $U(C_1) + E[U(C_2)]$.

- (a) Find the first-order condition for C_1 .
- (b) Show that $E[C_2] = C_1$ if Y_2 is not random or if utility is quadratic. (Hint: if utility is quadratic, $U'(C_2)$ is a linear function of C_2 , so $E[U'(C_2)] = U'(E[C_2])$).
- (c) Show that if $U'''(\bullet) > 0$ and Y_2 is random, $E[C_2] > C_1$. (Such saving due to uncertainty is known as *precautionary* saving. See Section 7.6.)
- (d) Suppose that the government marginally lowers τ_1 and raises τ_2 by the same amount, so that its expected total revenue, $\tau_1 Y_1 + \tau_2 E[Y_2]$, is unchanged. Implicitly differentiate the first-order condition in part (a) to find an expression for how C_1 responds to this change.
- (e) Show that C_1 is unaffected by this change if Y_2 is not random or if utility is quadratic.
- (f) Show that C_1 increases in response to this change if $U'''(\bullet) > 0$ and Y_2 is random.
- (g) If the utility function is constant-relative-risk-aversion, what is the sign of $U'''(\bullet)$?

2.13. Consider the Diamond model with logarithmic utility and Cobb-Douglas production. Describe how each of the following affects k_{t+1} as a function of k_t .

- (a) A rise in n .
- (b) A downward shift of the production function (that is, $f(k)$ takes the form Bk^α , and B falls).
- (c) A rise in α .

2.14. A discrete-time version of the Solow model. Suppose $Y_t = F(K_t, A_t L_t)$, with $F(\bullet)$ having constant returns to scale and the intensive form of the production function satisfying the Inada conditions. Suppose also that $A_{t+1} = (1 + g)A_t$, $L_{t+1} = (1 + n)L_t$, and $K_{t+1} = K_t + sY_t - \delta K_t$.

- (a) Find an expression for k_{t+1} as a function of k_t .
- (b) Sketch k_{t+1} as a function of k_t . Does the economy have a balanced growth path? If the initial level of k differs from the value on the balanced growth path, does the economy converge to the balanced growth path?
- (c) Find an expression for consumption per unit of effective labor on the balanced growth path as a function of the balanced-growth-path value of k . What is the marginal product of capital, $f'(k)$, when k maximizes consumption per unit of effective labor on the balanced growth path?

(d) Assume that the production function is Cobb–Douglas.

- (i) What is k_{t+1} as a function of k_t ?
- (ii) What is k^* , the value of k on the balanced growth path?
- (iii) Along the lines of equations (2.62)–(2.64), in the text, linearize the expression in subpart (i) around $k_t = k^*$, and find the rate of convergence of k to k^* .

2.15. Depreciation in the Diamond model and microeconomic foundations for the Solow model. Suppose that in the Diamond model capital depreciates at rate δ , so that $r_t = f'(k_t) - \delta$.

- (a) How, if at all, does this change in the model affect equation (2.60) giving k_{t+1} as a function of k_t ?
- (b) In the special case of logarithmic utility, Cobb–Douglas production, and $\delta = 1$, what is the equation for k_{t+1} as a function of k_t ? Compare this with the analogous expression for the discrete-time version of the Solow model with $\delta = 1$ from part (a) of Problem 2.14.

2.16. Social security in the Diamond model. Consider a Diamond economy where g is zero, production is Cobb–Douglas, and utility is logarithmic.

- (a) **Pay-as-you-go social security.** Suppose the government taxes each young individual amount T and uses the proceeds to pay benefits to old individuals; thus each old person receives $(1 + n)T$.
 - (i) How, if at all, does this change affect equation (2.61) giving k_{t+1} as a function of k_t ?
 - (ii) How, if at all, does this change affect the balanced-growth-path value of k ?
 - (iii) If the economy is initially on a balanced growth path that is dynamically efficient, how does a marginal increase in T affect the welfare of current and future generations? What happens if the initial balanced growth path is dynamically inefficient?
- (b) **Fully funded social security.** Suppose the government taxes each young person amount T and uses the proceeds to purchase capital. Individuals born at t therefore receive $(1 + r_{t-1})T$ when they are old.
 - (i) How, if at all, does this change affect equation (2.61) giving k_{t+1} as a function of k_t ?
 - (ii) How, if at all, does this change affect the balanced-growth-path value of k ?

2.17. The basic overlapping-generations model. (This follows Samuelson, 1958, and Allais, 1947.) Suppose, as in the Diamond model, that N_t 2-period-lived individuals are born in period t and that $N_t = (1 + n)N_{t-1}$. For simplicity, let utility be logarithmic with no discounting: $U_t = \ln(C_{1t}) + \ln(C_{2t+1})$.

The production side of the economy is simpler than in the Diamond model. Each individual born at time t is endowed with A units of the economy's single good. The good can either be consumed or stored. Each unit stored yields $x > 0$ units of the good in the following period.³⁷

³⁷Note that this is the same as the Diamond economy with $g = 0$, $F(K_t, AL_t) = AL_t + xK_t$.

Finally, assume that in the initial period, period 0, in addition to the N_0 young individuals each endowed with A units of the good, there are $[1/(1+n)]N_0$ individuals who are alive only in period 0. Each of these "old" individuals is endowed with some amount Z of the good; their utility is simply their consumption in the initial period, C_{20} .

- (a) Describe the decentralized equilibrium of this economy. (Hint: given the overlapping-generations structure, will the members of any generation engage in transactions with members of another generation?)
- (b) Consider paths where the fraction of agents' endowments that is stored, f_t , is constant over time. What is total consumption (that is, consumption of all the young plus consumption of all the old) per person on such a path as a function of f ? If $x < 1 + n$, what value of f satisfying $0 \leq f \leq 1$ maximizes consumption per person? Is the decentralized equilibrium Pareto-efficient in this case? If not, how can a social planner raise welfare?

2.18. Stationary monetary equilibria in the Samuelson overlapping-generations model. (Again this follows Samuelson, 1958.) Consider the setup described in Problem 2.17. Assume that $x < 1 + n$. Suppose that the old individuals in period 0, in addition to being endowed with Z units of the good, are each endowed with M units of a storable, divisible commodity, which we will call money. Money is not a source of utility.

- (a) Consider an individual born at t . Suppose the price of the good in units of money is P_t in t and P_{t+1} in $t + 1$. Thus the individual can sell units of endowment for P_t units of money and then use that money to buy P_t/P_{t+1} units of the next generation's endowment the following period. What is the individual's behavior as a function of P_t/P_{t+1} ?
- (b) Show that there is an equilibrium with $P_{t+1} = P_t/(1+n)$ for all $t \geq 0$ and no storage, and thus that the presence of "money" allows the economy to reach the golden-rule level of storage.
- (c) Show that there are also equilibria with $P_{t+1} = P_t/x$ for all $t \geq 0$.
- (d) Finally, explain why $P_t = \infty$ for all t (that is, money is worthless) is also an equilibrium. Explain why this is the *only* equilibrium if the economy ends at some date, as in Problem 2.19(b), below. (Hint: reason backward from the last period.)

2.19. The source of dynamic inefficiency. There are two ways in which the Diamond and Samuelson models differ from textbook models. First, markets are incomplete: because individuals cannot trade with individuals who have not been born, some possible transactions are ruled out. Second, because time goes on forever, there is an infinite number of agents. This problem asks you to investigate which of these is the source of the possibility of dynamic inefficiency. For simplicity, it focuses on the Samuelson overlapping-generations model (see the previous two problems), again with log utility and no discounting. To simplify further, it assumes $n = 0$ and $0 < x < 1$. The basic issues, however, are general.

and $\delta = 1$. With this production function, since individuals supply 1 unit of labor when they are young, an individual born in t obtains A units of the good. And each unit saved yields $1 + r = 1 + \partial F(K, AL)/\partial K - \delta = 1 + x - 1 = x$ units of second-period consumption.

- (a) **Incomplete markets.** Suppose we eliminate incomplete markets from the model by allowing all agents to trade in a competitive market "before" the beginning of time. That is, a Walrasian auctioneer calls out prices $Q_0, Q_1, Q_2, \dots$ for the good at each date. Individuals can then make sales and purchases at these prices given their endowments and their ability to store. The budget constraint of an individual born at t is thus $Q_t C_{1t} + Q_{t+1} C_{2t+1} = Q_t(A - S_t) + Q_{t+1}xS_t$, where S_t (which must satisfy $0 \leq S_t \leq A$) is the amount the individual stores.
- (i) Suppose the auctioneer announces $Q_{t+1} = Q_t/x$ for all $t > 0$. Show that in this case individuals are indifferent concerning how much to store, that there is a set of storage decisions such that markets clear at every date, and that this equilibrium is the same as the equilibrium described in part (a) of Problem 2.17.
- (ii) Suppose the auctioneer announces prices that fail to satisfy $Q_{t+1} = Q_t/x$ at some date. Show that at the first date that does not satisfy this condition the market for the good cannot clear, and thus that the proposed price path cannot be an equilibrium.
- (b) **Infinite duration.** Suppose that the economy ends at some date T . That is, suppose the individuals born at T live only one period (and hence seek to maximize C_{1T}), and that thereafter no individuals are born. Show that the decentralized equilibrium is Pareto-efficient.
- (c) In light of these answers, is it incomplete markets or infinite duration that is the source of dynamic inefficiency?

2.20. Explosive paths in the Samuelson overlapping-generations model (See Black, 1974; Brock, 1975; and Calvo, 1978a.) Consider the setup described in Problem 2.18. Assume that x is zero, and assume that utility is constant-relative-risk-aversion with $\theta < 1$ rather than logarithmic. Finally, assume for simplicity that $n = 0$.

- (a) What is the behavior of an individual born at t as a function of P_t/P_{t-1} ? Show that the amount of his or her endowment that the individual sells for money is an increasing function of P_t/P_{t-1} and approaches zero as this ratio approaches zero.
- (b) Suppose $P_0/P_1 < 1$. How much of the good are the individuals born in period 0 planning to buy in period 1 from the individuals born then? What must P_1/P_2 be for the individuals born in period 1 to want to supply this amount?
- (c) Iterating this reasoning forward, what is the qualitative behavior of P_t/P_{t+1} over time? Does this represent an equilibrium path for the economy?
- (d) Can there be an equilibrium path with $P_0/P_1 > 1$?

Chapter 3

BEYOND THE SOLOW MODEL: NEW GROWTH THEORY

The models we have seen so far do not provide satisfying answers to our central questions about economic growth. The models' principal result is a negative one: if capital's earnings reflect its contribution to output and if its share in total income is modest, then capital accumulation cannot account for a large part of either long-run growth or cross-country income differences. And the only determinant of income in the models other than capital is a mystery variable, the "effectiveness of labor" (A), whose exact meaning is not specified and whose behavior is taken as exogenous.

This chapter therefore investigates the fundamental questions of growth theory more deeply. It considers two broad views. The first view is that the driving force of growth is the accumulation of knowledge. This view agrees with the Solow model and the models of Chapter 2 that capital accumulation is not central to growth. But it differs from these models in explicitly interpreting the effectiveness of labor as knowledge and in formally modeling its evolution over time. This view is the subject of Part A of the chapter. We will analyze the dynamics of the economy when knowledge accumulation is modeled explicitly and consider various views concerning how knowledge is produced and what determines the allocation of resources to knowledge production.

The second view is that, contrary to the Solow model and the models of Chapter 2, capital is central to growth. Specifically, we will consider models that take a broader view of capital than we have considered so far—most importantly, extending it to include human capital. These models imply that physical capital's income share may not be a good guide to the overall importance of capital. We will see that, as a result, it is possible for capital accumulation alone to have large effects on real incomes. These models are the subject of Part B of the chapter.

The implied values of the parameters are $\alpha = 0.48(0.07)$, $\beta = 0.23(0.05)$, and $\lambda = 0.0142(0.0019)$. The estimates are broadly in line with the predictions of the model: countries converge toward their balanced growth paths at about the rate that the model predicts, and the estimated capital shares are broadly similar to what direct evidence suggests.

Overall, the evidence suggests that a model that maintains the assumption of diminishing returns to capital but that adopts a broader definition of capital than traditional physical capital, and therefore implies a total capital share closer to 1 than to $\frac{1}{3}$, provides a good first approximation to the cross-country data.

Problems

3.1. Consider the model of Section 3.2 with $\theta < 1$.

(a) On the balanced growth path, $\dot{A} = g_A^* A(t)$, where g_A^* is the balanced-growth-path value of g_A . Use this fact and equation (3.7) to derive an expression for $A(t)$ on the balanced growth path in terms of B , a_L , γ , θ , and $L(t)$.

(b) Use your answer to part (a) and the production function, (3.6), to obtain an expression for $Y(t)$ on the balanced growth path. Find the value of a_L that maximizes output on the balanced growth path.

3.2. Consider two economies (indexed by $i = 1, 2$) described by $Y_i(t) = K_i(t)^\theta$ and $\dot{K}_i(t) = s_i Y_i(t)$, where $\theta > 1$. Suppose that the two economies have the same initial value of K , but that $s_1 > s_2$. Show that Y_1/Y_2 is continually rising.

3.3. **Lags in a model of growth with an explicit knowledge-production sector.** Assume that final-goods production is given by $Y(t) = A(t)(1 - a_L)L$, where a_L is the fraction of the population engaged in knowledge production and L is population. a_L and L are exogenous and constant.

Suppose that knowledge becomes useful in generating new knowledge only with a lag, so that $\dot{A}(t) = Ba_L LJ(t)$, where J is the stock of knowledge useful in generating new knowledge. $J(t)$ is given by $\int_{\tau=0}^{\infty} (1 - e^{-v\tau}) \dot{A}(t - \tau) d\tau$, where $v > 0$. A simple way to check if there is a balanced growth path is to guess that it is possible for A to follow $A(t) = Ce^{gt}$, and look for candidate values of g .

(a) Show that the equation for $J(t)$ and the guess that $A(t) = Ce^{gt}$ imply $J(t) = [v/(v + g)]A(t)$.

(b) Is there any positive value of g such that, with $J(t)$ given by the expression in part (a) and $\dot{A}(t)$ given by $Ba_L LJ(t)$, $A(t)$ in fact follows $A(t) = Ce^{gt}$? How does that value of g depend on v ? What is its value as v approaches infinity? Explain intuitively how it is possible for a temporary delay in the availability of knowledge to permanently reduce the growth rate of the economy.

3.4. Consider the economy analyzed in Section 3.3. Assume that $\theta + \beta < 1$ and $n > 0$, and that the economy is on its balanced growth path. Describe how

each of the following changes affects the $\dot{g}_A = 0$ and $\dot{g}_K = 0$ lines and the position of the economy in (g_A, g_K) space at the moment of the change:

- (a) An increase in n .
- (b) An increase in a_K .
- (c) An increase in θ .

3.5. Consider the economy described in Section 3.3, and assume $\beta + \theta < 1$ and $n > 0$. Suppose the economy is initially on its balanced growth path, and that there is a permanent increase in s .

- (a) How, if at all, does the change affect the $\dot{g}_A = 0$ and $\dot{g}_K = 0$ loci? How, if at all, does it affect the location of the economy in (g_A, g_K) space at the time of the change?
- (b) What are the dynamics of g_A and g_K after the increase in s ? Sketch the path of log output per worker.
- (c) Intuitively, how does the effect of the increase in s compare with its effect in the Solow model?

3.6. Consider the model of Section 3.3 with $\beta + \theta = 1$ and $n = 0$.

- (a) Using (3.14) and (3.15), find the value that A/K must have for g_K and g_A to be equal.
- (b) Using your result in part (a), find the growth rate of A and K when $g_K = g_A$.
- (c) How does an increase in s affect the long-run growth rate of the economy?
- (d) What value of a_K maximizes the long-run growth rate of the economy? Intuitively, why is this value not increasing in β , the importance of capital in the R&D sector?

3.7. **Learning-by-doing.** Suppose that output is given by equation (3.22), $Y(t) = K(t)^\alpha [A(t)L(t)]^{1-\alpha}$, that L is constant and equal to 1, that $\dot{K}(t) = sY(t)$, and that knowledge accumulation occurs as a side effect of goods production: $\dot{A}(t) = BY(t)$.

- (a) Find expressions for $g_A(t)$ and $g_K(t)$ in terms of $A(t)$, $K(t)$, and the parameters.
- (b) Sketch the $\dot{g}_A = 0$ and $\dot{g}_K = 0$ lines in (g_A, g_K) space.
- (c) Does the economy converge to a balanced growth path? If so, what are the growth rates of K , A , and Y on the balanced growth path?
- (d) How does an increase in s affect long-run growth?

3.8. Suppose that output at firm i is given by $Y_i = K_i^\alpha L_i^{1-\alpha} [K^\phi L^{-\phi}]$, where K_i and L_i are the amounts of capital and labor used by the firm, K and L are the aggregate amounts of capital and labor, and $\alpha > 0$, $\phi > 0$, and $0 < \alpha + \phi < 1$. Assume that factors are paid their private marginal products; thus $r = \partial Y_i / \partial K_i$. Assume that the dynamics of K and L are given by $\dot{K} = sY$ and $\dot{L} = nL$, and that K_i / L_i is the same for all firms.

- (a) What is r as a function of K/L ?

- (b) What is K/L on the balanced growth path? What is r on the balanced growth path?
- (c) "If an increase in domestic saving raises domestic investment, positive externalities from capital would mitigate the decline in the private marginal product of capital. Thus the combination of positive externalities from capital and moderate barriers to capital mobility may be the source of Feldstein and Horioka's findings about saving and investment described in Chapter 1." Does your analysis in parts (a) and (b) support this claim? Explain intuitively.

3.9. (This follows Rebelo, 1991.) Assume that there are two factors of production, capital and land. Capital is used in both sectors, whereas land is used only in producing consumption goods. Specifically, the production functions are $C(t) = K_C(t)^\alpha R^{1-\alpha}$ and $\dot{K}(t) = BK_K(t)$, where K_C and K_K are the amounts of capital used in the two sectors (so $K_C(t) + K_K(t) = K(t)$) and R is the amount of land, and $0 < \alpha < 1$ and $B > 0$. Factors are paid their marginal products, and capital can move freely between the two sectors. R is normalized to 1 for simplicity.

- (a) Let $P_K(t)$ denote the price of capital goods relative to consumption goods at time t . Use the fact that the earnings of capital in units of consumption goods in the two sectors must be equal to derive a condition relating $P_K(t)$, $K_C(t)$, and the parameters α and B . If K_C is growing at rate $g_K(t)$, at what rate must P_K be growing (or falling)? Let $g_P(t)$ denote this growth rate.
- (b) The real interest rate in terms of consumption is $B + g_P(t)$.³⁵ Thus, assuming that households have our standard utility function, (3.28), the growth rate of consumption must be $(B + g_P - \rho)/\sigma \equiv g_C$. Assume $\rho < B$.
- (i) Use your results in part (a) to express $g_C(t)$ in terms of $g_K(t)$ rather than $g_P(t)$.
- (ii) Given the production function for consumption goods, at what rate must K_C be growing for C to be growing at rate $g_C(t)$?
- (iii) Combine your answers to (i) and (ii) to solve for $g_K(t)$ and $g_C(t)$ in terms of the underlying parameters.
- (c) Suppose that investment income is taxed at rate τ , so that the real interest rate households face is $(1 - \tau)(B + g_P)$. How, if at all, does τ affect the equilibrium growth rate of consumption?

3.10. (This follows Krugman, 1979; see also Grossman and Helpman, 1991b.) Suppose the world consists of two regions, the "North" and the "South." Output and capital accumulation in region i ($i = N, S$) are given by $Y_i(t) = K_i(t)^\alpha [A_i(t)(1 - a_{Li})L_i]^{1-\alpha}$, $\dot{K}_i(t) = s_i Y_i(t)$. New technologies are developed in the North. Specifically, $\dot{A}_N(t) = Ba_{LN}L_N A_N(t)$. Improvements in Southern technology, on the other hand, are made by learning from Northern technology: $\dot{A}_S(t) = \mu a_{LS}L_S[A_N(t) - A_S(t)]$ if $A_N(t) > A_S(t)$; otherwise $\dot{A}_S(t) = 0$. a_{LN}

³⁵To see this, note that capital in the investment sector produces new capital at rate B and changes in value relative to the consumption good at rate g_P . (Because the return to capital is the same in the two sectors, the same must be true of capital in the consumption sector.)

is the fraction of the Northern labor force engaged in R&D, and a_{LS} is the fraction of the Southern labor force engaged in learning Northern technology; the rest of the notation is standard. Note that L_N and L_S are assumed constant.

- (a) What is the long-run growth rate of Northern output per worker?
- (b) Define $Z(t) = A_S(t)/A_N(t)$. Find an expression for $\dot{Z}$ as a function of Z and the parameters of the model. Is Z stable? If so, what value does it converge to? What is the long-run growth rate of Southern output per worker?
- (c) Assume $a_{LN} = a_{LS}$ and $s_N = s_S$. What is the ratio of output per worker in the South to output per worker in the North when both economies have converged to their balanced growth paths?

3.11. Delays in the transmission of knowledge to poor countries.

- (a) Assume that the world consists of two regions, the North and the South. The North is described by $Y_N(t) = A_N(t)(1 - a_L)L_N$ and $\dot{A}_N(t) = a_L L_N A_N(t)$. The South does not do R&D but simply uses the technology developed in the North; however, the technology used in the South lags the North's by τ years. Thus $Y_S(t) = A_S(t)L_S$ and $A_S(t) = A_N(t - \tau)$. If the growth rate of output per worker in the North is 3 percent per year, and if a_L is close to 0, what must τ be for output per worker in the North to exceed that in the South by a factor of 10?
- (b) Suppose instead that both the North and the South are described by the Solow model: $y_i(t) = f(k_i(t))$, where $y_i(t) \equiv Y_i(t)/A_i(t)L_i(t)$ and $k_i(t) \equiv K_i(t)/A_i(t)L_i(t)$ ($i = N, S$). As in the Solow model, assume $\dot{K}_i(t) = sY_i(t) - \delta K_i(t)$ and $\dot{L}_i(t) = nL_i(t)$; the two countries are assumed to have the same saving rates and rates of population growth. Finally, $\dot{A}_N(t) = gA_N(t)$ and $A_S(t) = A_N(t - \tau)$.
 - (i) Show that the value of k on the balanced growth path, k^* , is the same for the two countries.
 - (ii) Does introducing capital change the answer to part (a)? Explain. (Continue to assume $g = 3\%$.)

3.12. Consider an economy described by the model of Part B of this chapter that is on its balanced growth path. Suppose there is a permanent increase in the rate of population growth. How does this affect output per worker over time?

3.13. Consider the model of Part B of the chapter.

- (a) What is consumption per unit of effective labor on the balanced growth path?
- (b) What values of s_K and s_H maximize this value?

3.14. Suppose that, despite the political obstacles, the United States permanently reduces its budget deficit from 3 percent of GDP to zero. Suppose that the economy is described by the model of Part B of the chapter, and that $\alpha = 0.35$ and $\beta = 0.4$. Suppose that initially $s_K = s_H = 0.15$, and that s_K rises by the full amount of the fall in the deficit.

- (a) By how much does output eventually rise relative to what it would have been without the deficit reduction?
- (b) By how much does consumption rise relative to what it would have been without the deficit reduction?
- (c) What is the immediate effect of the deficit reduction on consumption? About how long does it take for consumption to return to what it would have been without the deficit reduction?
- (d) Compare your results with the results of Problem 1.6.

3.15. Consider the following variant of our model with physical and human capital:

$$Y(t) = [(1 - a_K)K(t)]^\alpha [(1 - a_H)H(t)]^{1-\alpha}, \quad 0 < \alpha < 1, \quad 0 < a_K < 1, \quad 0 < a_H < 1,$$

$$\dot{K}(t) = sY(t) - \delta_K K(t),$$

$$\dot{H}(t) = B[a_K K(t)]^\gamma [a_H H(t)]^\phi [A(t)L(t)]^{1-\gamma-\phi} - \delta_H H(t), \quad \gamma > 0, \quad \phi > 0, \quad \gamma + \phi < 1,$$

$$\dot{L}(t) = nL(t),$$

$$\dot{A}(t) = gA(t),$$

where a_K and a_H are the fractions of the stocks of physical and human capital used in the education sector.

This model assumes that human capital is produced in its own sector with its own production function. Bodies (L) are useful only as something to be educated, not as an input into the production of final goods. Similarly, knowledge (A) is useful only as something that can be conveyed to students, not as a direct input into goods production.

- (a) Define $k = K/AL$ and $h = H/AL$. Derive equations for $\dot{k}$ and $\dot{h}$.
- (b) Find an equation describing the set of combinations of h and k such that $\dot{k} = 0$. Sketch in (h, k) space. Do the same for $\dot{h} = 0$.
- (c) Does this economy have a balanced growth path? If so, is it unique? Is it stable? What are the growth rates of output per person, physical capital per person, and human capital per person on the balanced growth path?
- (d) Suppose the economy is initially on a balanced growth path, and that there is a permanent increase in s . How does this change affect the path of output per person over time?

3.16. Use the production function, (3.43), and equations (3.55) and (3.56) to derive the expressions in equations (3.61) and (3.62) for the marginal products of physical and human capital on the balanced growth path of the model of Part B of this chapter.

3.17. **Constant returns to physical and human capital together.** Suppose the production function is $Y(t) = K(t)^\alpha H(t)^{1-\alpha}$ ($0 < \alpha < 1$), and that K and H evolve according to $\dot{K}(t) = s_K Y(t)$, $\dot{H}(t) = s_H Y(t)$.

- (a) Show that regardless of the initial levels of K and H (as long as both are positive), the ratio K/H converges to some balanced-growth-path level, $(K/H)^*$.

- (b) Once K/H has converged to $(K/H)^*$, what are the growth rates of K , H , and Y ?
- (c) How, if at all, does the growth rate of Y on the balanced growth path depend on s_K and s_H ?
- (d) Suppose K/H starts off at a level that is smaller than $(K/H)^*$. Is the initial growth rate of Y greater than, less than, or equal to its growth rate on the balanced growth path?

3.18. Increasing returns in a model with human capital. (This follows Lucas, 1988.) Suppose that $Y(t) = K(t)^\alpha [(1 - a_H)H(t)]^\beta$, $\dot{H}(t) = Ba_H H(t)$, and $\dot{K}(t) = sY(t)$, and assume $\alpha + \beta > 1$.³⁶

- (a) What is the growth rate of H ?
- (b) Does the economy converge to a balanced growth path? If so, what are the growth rates of K and Y on the balanced growth path?

3.19. The speed of convergence in the model of Part B of this chapter.

- (a) Use the production function, (3.48), and the equations of motion for k and h , (3.49) and (3.50), to find an expression for $d[\ln y(t)]/dt$.
- (b) Take a first-order Taylor approximation of the expression in part (a) in $\ln k$ and $\ln h$ around $\ln k = \ln k^*$, $\ln h = \ln h^*$.
- (c) Using the expressions for $\ln k^*$ and $\ln h^*$ in equations (3.55) and (3.56), show that the expression you obtained in part (b) can be simplified to yield (3.65) in the text.

³⁶Lucas's model differs from this formulation by letting a_H and s be endogenous and potentially time-varying, and by assuming that the social and private returns to human capital differ.

on incorporating real-business-cycle ingredients into our models, we would only be making it more difficult to identify the forces that actually drive economic fluctuations. At the other extreme, if the assumptions of, say, the Prescott model are approximately correct—which is also possible—then that model provides a parsimonious representation of the central features of most actual fluctuations. By insisting on complexity we would again be missing the essence of fluctuations. Similar comments apply to the issue of calibration versus traditional statistical procedures: if one approach is more informative than the other, pursuing both is costly.

It is of course possible that actual fluctuations are complicated and involve important elements of both real-business-cycle and Keynesian theories. Thus we cannot rule out the real-business-cycle view, the Keynesian view, or intermediate views of the sources and nature of aggregate fluctuations. As a result, macroeconomists have little choice but to make tentative judgments, based on the currently available models and evidence, about what lines of inquiry are most promising. And they must remain open to the possibility that those judgments will need to be revised.

Problems

- 4.1. Redo the calculations reported in Table 4.1 for any country other than the United States.
- 4.2. Redo the calculations reported in Table 4.3 for the following:⁴⁴
 - (a) Employees' compensation as a share of national income.
 - (b) The labor force participation rate.
 - (c) The federal government budget deficit as a share of GDP.
 - (d) The Standard and Poor 500 composite stock price index.
 - (e) The difference in yields between Moody's BAA and AAA bonds.
 - (f) The difference in yields between 10-year and 3-month U.S. Treasury securities.
 - (g) The weighted average exchange rate of the U.S. dollar against the currencies of other G-10 countries.
- 4.3. Let $\ln A_0$ denote the value of A in period 0, and let the behavior of $\ln A$ be given by equations (4.8)–(4.9).
 - (a) Express $\ln A_1$, $\ln A_2$, and $\ln A_3$ in terms of $\ln A_0$, ε_{A1} , ε_{A2} , ε_{A3} , $\bar{A}$, and g .
 - (b) In light of the fact that the expectations of the ε_A 's are zero, what are the expectations of $\ln A_1$, $\ln A_2$, and $\ln A_3$ given $\ln A_0$, $\bar{A}$, and g ?

⁴⁴Annual values for all of these series are published in the *Economic Report of the President*. Quarterly values are available from the Citibase data bank.

- 4.4. Suppose the period- t utility function, u_t , is $u_t = \ln c_t + b(1 - \ell_t)^{1-\gamma}/(1 - \gamma)$, $b > 0$, $\gamma > 0$, rather than by (4.7).
- (a) Consider the one-period problem analogous to that investigated in (4.12)–(4.15). How, if at all, does labor supply depend on the wage?
 - (b) Consider the two-period problem analogous to that investigated in (4.16)–(4.21). How does the relative demand for leisure in the two periods depend on the relative wage? How does it depend on the interest rate? Explain intuitively why γ affects the responsiveness of labor supply to wages and the interest rate.
- 4.5. Consider the problem investigated in (4.16)–(4.21).
- (a) Show that an increase in both w_1 and w_2 that leaves w_1/w_2 unchanged does not affect ℓ_1 or ℓ_2 .
 - (b) Now assume that the household has initial wealth of amount $Z > 0$.
 - (i) Does (4.23) continue to hold? Why or why not?
 - (ii) Does the result in (a) continue to hold? Why or why not?
- 4.6. Suppose an individual lives for two periods and has utility $\ln C_1 + \ln C_2$.
- (a) Suppose the individual has labor income of Y_1 in the first period of life and zero in the second period. Second-period consumption is thus $(1 + r)(Y_1 - C_1)$; r , the rate of return, is potentially random.
 - (i) Find the first-order condition for the individual's choice of C_1 .
 - (ii) Suppose r changes from being certain to being uncertain, without any change in $E[r]$. How, if at all, does C_1 respond to this change?
 - (b) Suppose the individual has labor income of zero in the first period and Y_2 in the second. Second-period consumption is thus $Y_2 - (1 + r)C_1$. Y_2 is certain; again, r may be random.
 - (i) Find the first-order condition for the individual's choice of C_1 .
 - (ii) Suppose r changes from being certain to being uncertain, without any change in $E[r]$. How, if at all, does C_1 respond to this change?
- 4.7. (a) Use an argument analogous to that used to derive equation (4.23) to show that household optimization requires $b/(1 - \ell_t) = e^{-\rho} E_t[w_t(1 + r_{t+1})b/[w_{t+1}(1 - \ell_{t+1})]]$.
- (b) Show that this condition is implied by (4.23) and (4.26). (Note that [4.26] must hold in every period.)
- 4.8. **A simplified real-business-cycle model with additive technology shocks.** (This follows Blanchard and Fischer, 1989, pp. 329–331.) Consider an economy consisting of a constant population of infinitely-lived individuals. The representative individual maximizes the expected value of $\sum_{t=0}^{\infty} u(C_t)/(1 + \rho)^t$, $\rho > 0$. The instantaneous utility function, $u(C_t)$, is $u(C_t) = C_t - \theta C_t^2$, $\theta > 0$. Assume that C is always in the range where $u'(C)$ is positive.
- Output is linear in capital, plus an additive disturbance: $Y_t = AK_t + e_t$. There is no depreciation; thus $K_{t+1} = K_t + Y_t - C_t$, and the interest rate is A .

Assume $A = \rho$. Finally, the disturbance follows a first-order autoregressive process: $e_t = \phi e_{t-1} + \varepsilon_t$, where $-1 < \phi < 1$ and where the ε_t 's are mean-zero, i.i.d. shocks.

- (a) Find the first-order condition (Euler equation) relating C_t and expectations of C_{t+1} .
- (b) Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma e_t$. Given this guess, what is K_{t+1} as a function of K_t and e_t ?
- (c) What values must the parameters α, β , and γ have for the first-order condition in part (a) to be satisfied for all values of K_t and e_t ?
- (d) What are the effects of a one-time shock to ε on the paths of Y, K , and C ?

4.9. A simplified real-business-cycle model with taste shocks. (This follows Blanchard and Fischer, 1989, p. 361.) Consider the setup in Problem 4.8. Assume, however, that the technological disturbances (the e 's) are absent, and that the instantaneous utility function is $u(C_t) = C_t - \theta(C_t + \nu_t)^2$. The ν 's are mean-zero, i.i.d. shocks.

- (a) Find the first-order condition (Euler equation) relating C_t and expectations of C_{t+1} .
- (b) Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma \nu_t$. Given this guess, what is K_{t+1} as a function of K_t and ν_t ?
- (c) What values must the parameters α, β , and γ have for the first-order condition in (a) to be satisfied for all values of K_t and ν_t ?
- (d) What are the effects of a one-time shock to ν on the paths of Y, K , and C ?

4.10. The balanced growth path of the model of Section 4.3. Consider the model of Section 4.3 without any shocks. Let y^*, k^*, c^* , and G^* denote the values of $Y/AL, K/AL, C/AL$, and G/AL on the balanced growth path; w^* the value of w/A ; ℓ^* the value of L/N ; and r^* the value of r .

- (a) Use equations (4.1)–(4.4), (4.23), and (4.26) and the fact that $y^*, k^*, c^*, w^*, \ell^*$, and r^* are constant on the balanced growth path to find six equations in these six variables. (Hint: the fact that c in (4.23) is consumption per person, C/N , and c^* is the balanced-growth-path value of consumption per unit of effective labor, C/AL , implies that $c = c^* \ell^* A$ on the balanced growth path.)
- (b) Consider the parameter values assumed in Section 4.7. What are the implied shares of consumption and investment in output on the balanced growth path? What is the implied ratio of capital to annual output on the balanced growth path?

4.11. Solving a real-business-cycle model by finding the social optimum.⁴⁵ Consider the model of Section 4.5. Assume for simplicity that $n = g = \bar{A} = \bar{N} = 0$. Let $V(K_t, A_t)$, the *value function*, be the expected present value from the cur-

⁴⁵This problem uses dynamic programming and the method of undetermined coefficients. These two methods are explained in Section 10.4 and Section 4.6, respectively.

rent period forward of lifetime utility of the representative individual as a function of the capital stock and technology.

(a) Explain intuitively why $V(\bullet)$ must satisfy

$$V(K_t, A_t) = \max_{C_t, \ell_t} \{[\ln C_t + b \ln(1 - \ell_t)] + e^{-\rho} E_t[V(K_{t+1}, A_{t+1})]\}.$$

This condition is known as the *Bellman equation*.

Given the log-linear structure of the model, let us guess that $V(\bullet)$ takes the form $V(K_t, A_t) = \beta_0 + \beta_K \ln K_t + \beta_A \ln A_t$, where the values of the β 's are to be determined. Substituting this conjectured form and the facts that $K_{t+1} = Y_t - C_t$ and $E_t[\ln A_{t+1}] = \rho_A \ln A_t$ into the Bellman equation yields

$$V(K_t, A_t) = \max_{C_t, \ell_t} \{[\ln C_t + b \ln(1 - \ell_t)] + e^{-\rho} [\beta_0 + \beta_K \ln(Y_t - C_t) + \beta_A \rho_A \ln A_t]\}.$$

- (b) Find the first-order condition for C_t . Show that it implies that C_t/Y_t does not depend on K_t or A_t .
- (c) Find the first-order condition for ℓ_t . Use this condition and the result in part (b) to show that ℓ_t does not depend on K_t or A_t .
- (d) Substitute the production function and the results in parts (b) and (c) for the optimal C_t and ℓ_t into the equation above for $V(\bullet)$, and show that the resulting expression has the form $V(K_t, A_t) = \beta'_0 + \beta'_K \ln K_t + \beta'_A \ln A_t$.
- (e) What must β_K and β_A be so that $\beta'_K = \beta_K$ and $\beta'_A = \beta_A$?⁴⁶
- (f) What are the implied values of C/Y and ℓ ? Are they the same as those found in Section 4.5 for the case of $n = g = 0$?

4.12. Suppose the behavior of technology is described by some process other than (4.8)–(4.9). Do $s_t = \hat{s}$ and $\ell_t = \hat{\ell}$ for all t continue to solve the model of Section 4.5? Why or why not?

4.13. Consider the model of Section 4.5. Suppose, however, that the instantaneous utility function, u_t , is given by $u_t = \ln c_t + b(1 - \ell_t)^{1-\gamma}/(1 - \gamma)$, $b > 0$, $\gamma > 0$, rather than by (4.7) (see Problem 4.4).

- (a) Find the first-order condition analogous to equation (4.26) that relates current leisure and consumption given the wage.
- (b) With this change in the model, is the saving rate (s) still constant?
- (c) Is leisure per person ($1 - \ell$) still constant?

4.14. (a) If the $\tilde{A}_t$'s are uniformly zero and if $\ln Y_t$ evolves according to (4.39), what path does $\ln Y_t$ settle down to? (Hint: note that we can rewrite (4.39) as $\ln Y_t - (n + g)t = Q + \alpha[\ln Y_{t-1} - (n + g)(t - 1)] + (1 - \alpha)\tilde{A}_t$, where $Q \equiv \alpha \ln \hat{s} + (1 - \alpha)[\bar{A} + \ln \hat{\ell} + \bar{N}] - \alpha(n + g)$.)

(b) Defining $\tilde{Y}_t$ as the difference between $\ln Y_t$ and the path found in (a), derive (4.40).

⁴⁶The calculation of β_0 is tedious and is therefore omitted.

4.15. The derivation of the log-linearized equation of motion for capital, (4.52). Consider the equation of motion for capital, $K_{t+1} = K_t + K_t^\alpha (A_t L_t)^{1-\alpha} - C_t - G_t - \delta K_t$.

- (a) (i) Show that $\partial \ln K_{t+1} / \partial \ln K_t$ (holding A_t, L_t, C_t , and G_t fixed) is $(1 + r_{t+1})(K_t / K_{t-1})$.
 (ii) Show that this implies that $\partial \ln K_{t+1} / \partial \ln K_t$ evaluated at the balanced growth path is $(1 + r^*)/e^{n+g}$.⁴⁷
 (b) Show that

$$\tilde{K}_{t+1} \approx \lambda_1 \tilde{K}_t + \lambda_2 (\tilde{A}_t + \tilde{L}_t) + \lambda_3 \tilde{G}_t + (1 - \lambda_1 - \lambda_2 - \lambda_3) \tilde{C}_t,$$

where $\lambda_1 \equiv (1 + r^*)/e^{n+g}$, $\lambda_2 \equiv (1 - \alpha)(r^* + \delta)/\alpha e^{n+g}$, and $\lambda_3 = -(r^* + \delta)(G/Y)^*/\alpha e^{n+g}$; and where $(G/Y)^*$ denotes the ratio of G to Y on the balanced growth path without shocks. (Hints: 1. Since the production function is Cobb-Douglas, $Y^* = (r^* + \delta)K^*/\alpha$; 2. On the balanced growth path, $K_{t-1} = e^{n+g} K_t$, which implies that $C^* = Y^* - G^* - \delta K^* - (e^{n+g} - 1)K^*$.)

- (c) Use the result in (b) and equations (4.43)-(4.44) to derive (4.52), where $b_{KK} = \lambda_1 + \lambda_2 a_{LK} + (1 - \lambda_1 - \lambda_2 - \lambda_3) a_{CK}$, $b_{KA} = \lambda_2 (1 + a_{LA}) + (1 - \lambda_1 - \lambda_2 - \lambda_3) a_{CA}$, and $b_{KG} = \lambda_2 a_{LG} + \lambda_3 + (1 - \lambda_1 - \lambda_2 - \lambda_3) a_{CG}$.

4.16. A Monte Carlo experiment, and the source of bias in OLS estimates of equation (4.56). Suppose output growth is described simply by $\Delta \ln y_t = \varepsilon_t$, where the ε 's are independent, mean-zero disturbances. Normalize the initial value of $\ln y$, $\ln y_0$, to zero for simplicity. This problem asks you to consider what occurs in this situation if one estimates equation (4.56), $\Delta \ln y_t = \alpha' + b \ln y_{t-1} + \varepsilon_t$, by ordinary least squares.

- (a) Suppose the sample size is 3, and suppose each ε is equal to 1 with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$. For each of the eight possible realizations of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ $((1, 1, 1), (1, 1, -1)$, and so on), what is the OLS estimate of b ? What is the average of the estimates? Explain intuitively why the estimates differ systematically from the true value of $b = 0$.
 (b) Suppose the sample size is 200, and suppose each ε is normally distributed with a mean of 0 and a variance of 1. Using a random-number generator on a computer, generate 200 such ε 's; then generate $\ln y$'s using $\Delta \ln y_t = \varepsilon_t$ and $\ln y_0 = 0$; then estimate (4.56) by OLS; finally, record the estimate of b . Repeat this process 500 times. What is the average estimate of b ? What fraction of the estimated b 's are negative?

⁴⁷One could express r^* in terms of the discount rate ρ . Campbell (1994) argues, however, that it is easier to discuss the model's implications in terms of r^* instead of ρ .

Chapter 5

TRADITIONAL KEYNESIAN THEORIES OF FLUCTUATIONS

5.1 Introduction

This chapter and the next develop models of fluctuations based on the assumption that there are barriers to the instantaneous adjustment of nominal prices and wages. As we will see, sluggish nominal adjustment causes changes in the aggregate demand for goods at a given level of prices to affect the amount that firms produce. As a result, it causes purely monetary disturbances (which affect only demand) to change employment and output. In addition, many real shocks, including changes in government purchases, investment demand, and technology, affect aggregate demand at a given price level; thus sluggish price adjustment creates a channel other than the intertemporal-substitution mechanism of real-business-cycle models through which these shocks affect employment and output.

This chapter takes nominal stickiness as given. It has two main goals. The first is to investigate aggregate demand. We will examine the determinants of aggregate demand at a given price level and the effects of changes in the price level. The second is to consider alternative assumptions about the form of nominal rigidity. We will investigate different assumptions' implications for firms' willingness to change output in response to changes in aggregate demand and for the behavior of real wages, markups, and inflation. Chapter 6 then turns to the questions of why nominal prices and wages might not adjust immediately to disturbances.

The Keynesian Approach to Modeling

As will quickly become clear, Keynesian models differ from real-business-cycle models not just in substance, but also in style. Real-business-cycle models typically begin with microeconomic assumptions about households' preferences, firms' production functions, the structure of markets, and the evolution of quantities over time. Thus the models are fully specified

operations are associated with changes in nominal interest rates. Given the discrete nature of the open-market operations and the specifics of how their timing is determined, it is not plausible that they occur endogenously at times when interest rates would have moved in any event. And although the issue has not been investigated formally, the fact that monetary expansions lower nominal rates strongly suggests that the changes in nominal rates represent changes in real rates as well. For example, monetary expansions lower nominal interest rates for terms as short as a day; it seems unlikely that they reduce expected inflation over such horizons.²⁸ Since real and Keynesian theories agree that changes in real rates affect real behavior, this evidence suggests that monetary changes have real effects.

Similarly, the exchange-rate regime appears to affect the behavior of real exchange rates. Under a fixed exchange rate, the central bank adjusts the money supply to keep the nominal exchange rate constant; under a floating exchange rate, it does not. There is strong evidence that not just nominal but also real exchange rates are much less volatile under fixed than floating exchange rates. In addition, when a central bank switches from pegging the nominal exchange rate against one currency to pegging the nominal exchange rate against another, the volatility of the two associated real exchange rates seems to change sharply as well. (See, for example, Genberg, 1978; Stockman, 1983; Mussa, 1986; and Baxter and Stockman, 1989). Since shifts between exchange-rate regimes are usually discrete, explaining this behavior of real exchange rates without appealing to real effects of monetary forces appears to require positing sudden large changes in the real shocks affecting economies. And again, both real and Keynesian theories predict that the behavior of real exchange rates has real effects.

The most significant limitation of this evidence is that the importance of these apparent effects of monetary changes on real interest rates and real exchange rates for quantities has not been determined. Baxter and Stockman (1989), for example, do not find any clear difference in the behavior of economic aggregates under floating and fixed exchange rates. Since real-business-cycle theories attribute fairly large changes in quantities to relatively modest movements in relative prices, however, a finding that the price changes were not important would be puzzling from the perspective of both real and Keynesian theories.

Problems

5.1. Consider the *IS-LM* model presented in Section 5.2. In this model, what are di/dM and dY/dM for a given value of P ?

²⁸Barro (1989) presents a model where monetary expansions lower expected inflation. The model requires that prices jump instantaneously in response to the expansions, however.

- 5.2. The derivation of the *LM* curve assumes that M is exogenous. But suppose instead that the Federal Reserve has some target interest rate $\bar{i}$ and that it adjusts M to keep i always equal to $\bar{i}$.
- (a) With this policy, what is the slope of the “*LM* curve” (that is, the set of combinations of i and Y that cause money demand and supply to be equal)?
 - (b) With this policy, what is the slope of the *AD* curve?
- 5.3. The government budget in the standard Keynesian model.
- (a) **The balanced budget multiplier.** (See Haavelmo, 1945.) Suppose that planned expenditure is given by (5.5), $E = C(Y - T) + I(i - \pi^e) + G$.
 - (i) How do equal increases in G and T affect the position of the *IS* curve? Specifically, what is the effect on Y for a given level of i ?
 - (ii) How do equal increases in G and T affect the position of the *AD* curve? Specifically, what is the effect on Y for a given level of P ?
 - (b) **Automatic stabilizers.** Suppose that tax revenues, T , instead of being exogenous, are a function of income: $T = T(Y)$, $T'(Y) > 0$. With this change, find how an increase in $T'(Y)$ affects the following:
 - (i) The slope of the *IS* curve.
 - (ii) The effects of changes in G and M on Y for a given P .
- 5.4. **The liquidity trap and the Pigou effect.** Assume that the nominal interest rate is so low that the opportunity cost of holding money is negligible. Suppose that as a result people are indifferent concerning the division of their wealth between money and other assets, and that they are therefore willing to change their money holdings without any change in the interest rate.
- (a) **The liquidity trap.** (Keynes, 1936.) In this situation, what is the slope of the *AD* curve? If prices are completely flexible (so the *AS* curve is vertical), is aggregate demand irrelevant to output?
 - (b) **The Pigou effect.** (Pigou, 1943.) Suppose that, in addition, planned expenditure depends on real wealth as well as the variables in (5.4). Since the public's holdings of high-powered money are one component of wealth, a fall in the price level increases real wealth. If prices are completely flexible (so the *AS* curve is vertical), is aggregate demand irrelevant to output?
- 5.5. **The Mundell effect.** (Mundell, 1963.) In the *IS-LM* model, how does a fall in expected inflation, π^e , affect i , Y , and $i - \pi^e$?
- 5.6. **The multiplier-accelerator.** (Samuelson, 1939.) Consider the following model of income determination. (1) Consumption depends on the previous period's income: $C_t = a + bY_{t-1}$. (2) The desired capital stock (or inventory stock) is proportional to the previous period's output: $K_t^* = cY_{t-1}$. (3) Investment equals the difference between the desired capital stock and the stock inherited from the previous period: $I_t = K_t^* - K_{t-1} = K_t^* - cY_{t-2}$. (4) Government purchases are constant: $G_t = \bar{G}$. (5) $Y_t = C_t + I_t + G_t$.
- (a) Express Y_t in terms of Y_{t-1} , Y_{t-2} , and the parameters of the model.
 - (b) Suppose $b = 0.9$ and $c = 0.5$. Suppose there is a one-time disturbance to government purchases; specifically, suppose that G is equal to $\bar{G} + 1$ in

period t and is equal to $\bar{G}$ in all other periods. How does this shock affect output over time?

- 5.7. (This follows Mankiw and Summers, 1986.) Suppose that the demand for real money balances depends on the interest rate, i , and on *disposable* income $Y - T$; in other words, suppose that the correct way to write the *LM* equation is $M/P = L(i, Y - T)$.
- (a) With this change to the *IS-LM-AS* model, can one tell whether a tax cut (that is, a fall in T) increases or decreases output? Assume a closed economy.
 - (b) Redo part (a) assuming an open economy under the assumptions that the exchange rate is floating, exchange-rate expectations are static, and capital is perfectly mobile.
 - (c) Redo part (b) assuming a fixed exchange rate.
- 5.8. Describe how each of the following changes affect income, the exchange rate, and net exports at a given price level under: (1) a floating exchange rate and perfect capital mobility, (2) a fixed exchange rate and perfect capital mobility, and (3) a floating exchange rate and imperfect capital mobility. Assume static exchange-rate expectations, and assume that planned expenditure is given by the expression in n. 8.
- (a) The demand for money at a given i and Y falls.
 - (b) The foreign interest rate rises.
 - (c) The country adopts protectionist policies, so that net exports at a given real exchange rate are higher than before.
- 5.9. **Exchange-market intervention.** Suppose that the central bank intervenes in the foreign exchange market by purchasing foreign currency for dollars, and that it *sterilizes* this intervention by selling bonds for dollars to keep the money stock unchanged. With this intervention, NX and CF must sum to a positive amount rather than to zero (see equation [5.21]).
- (a) What are the effects of this intervention on output, the exchange rate, and the price level under a floating exchange rate, static exchange-rate expectations, and imperfect capital mobility?
 - (b) How, if at all, do the results in part (a) change if capital is perfectly mobile?
- 5.10. **The algebra of exchange-rate overshooting.** Consider a simplified open-economy model: $m - p = hy - ki$, $y = b(\varepsilon - p) - a(i - \dot{p})$, $i = \dot{\varepsilon}$, $\dot{p} = \theta y$. The variables y , m , p , and ε are the logs of output, money, the price level, and the exchange rate, respectively; i is the nominal interest rate, and $\dot{p}$ is inflation. All variables are expressed as deviations from their usual values; p^* and i^* are normalized to zero, and are therefore omitted. The main changes from our usual model are that price adjustment takes a particularly simple form and that the equations are linear. h , k , b , a , and θ are all positive.
- Assume that initially $y = i = \dot{p} = m = p = 0$. Now suppose that there is a permanent increase in m .

- (a) Show that once prices have adjusted fully (so $\dot{p} = 0$), $y = i = 0$ and $p = \varepsilon = m$.
- (b) Show that there are parameter values such that at the time of the increase in m , ε jumps immediately to exactly m and then remains constant—so that there is neither overshooting nor undershooting.²⁹
- 5.11.** Consider the model of aggregate demand in an open economy with imperfect capital mobility in Section 5.3, without the simplification assumed in equation (5.22). In addition to our usual assumptions, assume $NX_{\varepsilon P^* / P} \geq E_{\varepsilon P^* / P}$, $NX_{i - \pi^e} \geq 0$, $NX_Y \leq 0$, and $E_Y - NX_Y < 1$.
- (a) Derive an expression for the slope of the IS^{**} curve (that is, the combinations of i and Y associated with the (i, Y, ε) combinations that solve [5.12] and [5.21]).
- (b) Does ε rise, fall, or remain constant as we move down the IS^{**} curve?
- (c) Is it still true that greater capital mobility (that is, a larger value of $CF'(\bullet)$) makes the IS^{**} curve flatter?
- 5.12.** The analysis of Case 1 in Section 5.4 assumes that employment is determined by labor demand. A more realistic assumption may be that employment at a given real wage equals the minimum of demand and supply; this is known as the *short-side rule*.
- (a) Draw diagrams showing the situation in the labor market under this assumption when
- P is at the level that generates the maximum possible output.
 - P is above the level that generates the maximum possible output.
- (b) With this assumption, what does the aggregate supply curve look like?
- 5.13.** Consider the model of aggregate supply in Case 2 of Section 5.4. Suppose that aggregate demand at $\bar{P}$ equals Y^{MAX} . Show the resulting situation in the labor market.
- 5.14.** Suppose that the production function is $Y = AF(L)$ (where $F'(\bullet) > 0$, $F''(\bullet) < 0$, and $A > 0$), and that A falls. How does this negative technology shock affect the AS curve under each of the models of aggregate supply in Section 5.4?
- 5.15. Destabilizing price flexibility.** (De Long and Summers, 1986b.) Consider the following closed-economy variant of the model in Problem 5.10: $y = -a(i - \dot{p})$, $m - p = -ki$, $\dot{p} = \theta y$. Assume $a > 0$, $k > 0$, $\theta > 0$, and $a\theta < 1$.
- (a) Assume that initially $y = i = \dot{p} = m = p = 0$. Now suppose that at some time—time 0 for convenience—there is a permanent drop in m to some lower level, m' .

²⁹The result that there are parameter values such that the exchange rate neither overshoots nor undershoots in response to a monetary disturbance implies that, except in unusual cases, there are perturbations of these parameter values that lead to each result. Showing this is complicated, however, and is therefore omitted.

- (i) What are the values of y and i at time 0? (Note that p cannot jump at the time of the change.) How does an increase in θ , the speed of price adjustment, affect $y(0)$? Explain intuitively.
 - (ii) What is the path of y after time 0?
 - (b) Suppose we measure the total amount of output volatility caused by the change in m as $V = \int_{t=0}^{\infty} y(t)^2 dt$. How is V affected by an increase in the speed of price adjustment, θ ?
- 5.16. Redo the regression reported in equation (5.40):
- (a) Incorporating more recent data.
 - (b) Incorporating more recent data, and using $M2$ rather than $M1$.

Chapter 6

MICROECONOMIC FOUNDATIONS OF INCOMPLETE NOMINAL ADJUSTMENT

This chapter is concerned with the microeconomic foundations of sluggish adjustment of nominal prices and wages. This subject is important for two reasons. First, it is central to Keynesian models. One of the models' main predictions is that monetary shocks have real effects, and the critical feature of the models that gives rise to this prediction is the presence of sluggish nominal adjustment. But, as described in the concluding section of the previous chapter, the evidence concerning whether monetary shocks have important real effects is controversial; thus the relevance of Keynesian models is not clear. One way to shed light on this issue is to investigate what microeconomic conditions are needed for nominal stickiness to arise. For example, some critics of traditional Keynesian models argue that the models' assumptions about price stickiness are inconsistent with any reasonable model of microeconomic behavior; they therefore conclude that microeconomic theory provides a strong case against the models' relevance. More generally, if the conditions needed for nominal stickiness appear implausible or inconsistent with microeconomic evidence, this would suggest that gradual nominal adjustment is unlikely to be important. If the needed conditions appear realistic, on the other hand, this would support the importance of nominal stickiness.

Second, the nature of incomplete nominal adjustment is important for policy. For example, we will see that if monetary shocks have real effects for the reasons described by the Lucas imperfect-information model (which is presented in Part A of the chapter), systematic feedback rules from economic developments to monetary policy have no effect on the real economy. Similarly, if nominal prices and wages are fully flexible, monetary policy is

reflect gradual progress in our understanding of the economy. But a theory that is so flexible that it cannot be contradicted by any set of observations is devoid of content. Thus if Keynesian theory is to be useful, there must be some questions about which it delivers clear predictions.

One view is that the central issue on which Keynesian theory must stand or fall is the real effects of nominal disturbances. A central element of all Keynesian models is that nominal prices or wages do not adjust immediately. As a result, the models predict that independent monetary disturbances affect real activity. If this prediction is contradicted by the data, it appears that the models would have to be abandoned rather than modified, and that the study of fluctuations would have to pursue the real-business-cycle models of Chapter 4 or the real non-Walrasian theories of the previous section. If this view of the defining element of Keynesian theory is right, evaluating and extending the evidence described in Section 5.6 concerning the effects of monetary shocks is critical to business-cycle research.

Problems

6.1. Consider the problem facing an individual in the Lucas model when P_i/P is unknown. The individual chooses L_i to maximize the expectation of U_i ; U_i continues to be given by equation (6.3).

- (a) Find the first-order condition for L_i , and rearrange it to obtain an expression for L_i in terms of $E[P_i/P]$. Take logs of this expression to obtain an expression for ℓ_i .
- (b) How does the amount of labor the individual supplies if he or she follows the certainty-equivalence rule in (6.17) compare with the optimal amount derived in part (a)? (Hint: how does $E[\ln(P_i/P)]$ compare with $\ln(E[P_i/P])$?)
- (c) Suppose that (as in the Lucas model) $\ln(P_i/P) = E[\ln(P_i/P) | P_i] + u_i$, where u_i is normal with a mean of zero and a variance that is independent of P_i . Show that this implies that $\ln\{E[(P_i/P) | P_i]\} = E[\ln(P_i/P) | P_i] + C$, where C is a constant whose value is independent of P_i . (Hint: note that $P_i/P = \exp\{E[\ln(P_i/P) | P_i]\} \exp\{u_i\}$, and show that this implies that the ℓ_i that maximizes expected utility differs from the certainty-equivalence rule in (6.17) only by a constant.)

6.2. (This follows Dixit and Stiglitz, 1977.) Suppose that the consumption index C_i in equation (6.2) is $C_i = [\int_{j=0}^1 Z_j^{1/\eta} C_{ij}^{(\eta-1)/\eta} dj]^{1/(\eta-1)}$, where C_{ij} is the individual's consumption of good j and Z_j is the taste shock for good j . Suppose the individual has amount Y_i to spend on goods. Thus the budget constraint is $\int_{j=0}^1 P_j C_{ij} dj = Y_i$.

- (a) Find the first-order condition for the problem of maximizing C_i subject to the budget constraint. Solve for C_{ij} in terms of Z_j , P_j , and the Lagrange multiplier on the budget constraint.
- (b) Use the budget constraint to find C_{ij} in terms of Z_j , P_j , Y_i , and the Z 's and P 's.

- (c) Substitute your result in part (b) into the expression for C_i and show that $C_i = Y_i/P$, where $P \equiv [\int_{j=0}^1 Z_j P_j^{1-\eta} dj]^{1/(1-\eta)}$.
- (d) Use the results in part (b) and part (c) to show that $C_{ij} = Z_j (P_j/P)^{-\eta} (Y_i/P)$.
- (e) Compare your results with (6.7) and (6.9) in the text.

6.3. Observational equivalence. (Sargent, 1976.) Suppose that the money supply is determined by $m_t = c'z_{t-1} + e_t$, where c and z are vectors and e_t is an i.i.d. disturbance uncorrelated with z_{t-1} . e_t is unpredictable and unobservable. Thus the expected component of m_t is $c'z_{t-1}$, and the unexpected component is e_t . In setting the money supply, the Federal Reserve responds only to variables that matter for real activity; that is, the variables in z directly affect y .

Now consider the following two models: (i) Only unexpected money matters, so $y_t = a'z_{t-1} + be_t + v_t$; (ii) all money matters, so $y_t = a'z_{t-1} + \beta m_t + v_t$. In each specification, the disturbance is i.i.d. and uncorrelated with z_{t-1} and e_t .

- (a) Is it possible to distinguish between these two theories? That is, given a candidate set of parameter values under, say, Model (i), are there parameter values under Model (ii) that have the same predictions? Explain.
- (b) Suppose that the Federal Reserve also responds to some variables that do not directly affect output; that is, suppose $m_t = c'z_{t-1} + \gamma'w_{t-1} + e_t$ and that Models (i) and (ii) are as before (with their disturbances now uncorrelated with w_{t-1} as well as with z_{t-1} and e_t). In this case, is it possible to distinguish between the two theories? Explain.

6.4. Suppose the economy is described by the model of Section 6.6. Assume, however, that P is the price index described in part (c) of Problem 6.2 (with all the Z_j 's equal to 1 for simplicity). In addition, assume that money-market equilibrium requires that total spending in the economy equal M . With these changes, is it still the case that in equilibrium, output of each good is given by (6.46) and that the price of each good is given by (6.47)?

6.5. Indexation. (See Gray, 1976, 1978, and Fischer, 1977b. This problem follows Ball, 1988.) Suppose production at firm i is given by $Y_i = SL_i^\alpha$, where S is a supply shock and $0 < \alpha \leq 1$. Thus in logs, $y_i = s + \alpha \ell_i$. Prices are flexible; thus (setting the constant term to zero for simplicity), $p_i = w_i + (1 - \alpha)\ell_i - s$. Aggregating the output and price equations yields $y = s + \alpha \ell$ and $p = w + (1 - \alpha)\ell - s$. Wages are partially indexed to prices: $w = \theta p$, where $0 \leq \theta \leq 1$. Finally, aggregate demand is given by $y = m - p$. s and m are independent, mean-zero random variables with variances V_s and V_m .

- (a) What are p, y, ℓ , and w as functions of m and s and the parameters α and θ ? How does indexation affect the response of employment to monetary shocks? How does it affect the response to supply shocks?
- (b) What value of θ minimizes the variance of employment?
- (c) Suppose the demand for a single firm's output is $y_i = y - \eta(p_i - p)$. Suppose all firms other than firm i index their wages to the price level by $w = \theta p$ as before, but that firm i indexes its wage to the price level by $w_i = \theta_i p$. Firm i continues to set its price as $p_i = w_i + (1 - \alpha)\ell_i - s$. The production function and the pricing equation then imply that $y_i = y - \phi(w_i - w)$, where $\phi \equiv \alpha\eta/[\alpha + (1 - \alpha)\eta]$.

- (i) What is employment at firm i , ℓ_i , as a function of $m, s, \alpha, \eta, \theta$, and θ_i ?
 - (ii) What value of θ_i minimizes the variance of ℓ_i ?
 - (iii) Find the Nash equilibrium value of θ . That is, find the value of θ such that if aggregate indexation is given by θ , the representative firm minimizes the variance of ℓ_i by setting $\theta_i = \theta$. Compare this value with the value found in part (b).
- 6.6. Synchronized price-setting.** Consider the Taylor model. Suppose, however, that every other period all of the individuals set their prices for that period and the next. That is, in period t prices are set for t and $t + 1$; in $t + 1$, no prices are set; in $t + 2$, prices are set for $t + 2$ and $t + 3$; and so on. As in the Taylor model, prices are both predetermined and fixed, and individuals set their prices according to (6.60). Finally, assume that m follows a random walk.
- (a) What is the representative individual's price in period t , x_t , as a function of m_t , $E_t m_{t+1}$, p_t , and $E_t p_{t-1}$?
 - (b) Use the fact that synchronization implies that p_t and p_{t+1} are both equal to x_t to solve for x_t in terms of m_t and $E_t m_{t+1}$.
 - (c) What are y_t and y_{t+1} ? Does the central result of the Taylor model—that nominal disturbances continue to have real effects after all prices have been changed—still hold? Explain intuitively.
- 6.7. The Fischer model with unbalanced price-setting.** Suppose the economy is as described by the model of Section 6.7, except that instead of half of the individuals setting their prices each period, fraction f set their prices in odd periods and fraction $1 - f$ set their prices in even periods. Thus the price level is $f p_t^1 + (1 - f) p_t^2$ if t is even and $(1 - f) p_t^1 + f p_t^2$ if t is odd. Derive expressions analogous to (6.57) and (6.58) for p_t and y_t for even and odd periods.
- 6.8. The instability of staggered price-setting.** (See Fethke and Policano, 1986; Ball and Cecchetti, 1988; and Ball and D. Romer, 1989.) Suppose the economy is described as in Problem 6.7, and assume for simplicity that m is a random walk (so $m_t = m_{t-1} + u_t$, where u is white noise and has a constant variance). Assume that the amount of profits an individual loses over two periods relative to always having $p_i = p_i^*$ is proportional to $(p_{it} - p_{it}^*)^2 + (p_{it+1} - p_{it+1}^*)^2$. If $f < \frac{1}{2}$ and $\phi < 1$, is the expected value of this loss larger for the individuals who set their prices in odd periods or for the individuals who set their prices in even periods? In light of this, would you expect to see staggered price-setting if $\phi < 1$?
- 6.9.** Consider the Taylor model with the money stock white noise rather than a random walk; that is, $m_t = \varepsilon_t$, where ε_t is serially uncorrelated. Solve the model using the method of undetermined coefficients. (Hint: in the equation analogous to (6.63), is it still reasonable to impose $\lambda + \nu = 1$?)
- 6.10.** Repeat Problem 6.9 using lag operators.
- 6.11.** (This follows Ball, 1994a.) Consider a continuous-time version of the Taylor model, so that $p(t) = (1/T) \int_{\tau=0}^T x(t - \tau) d\tau$, where T is the interval between each individual's price changes and $x(t - \tau)$ is the price set by individuals

who set their prices at time $t - \tau$. Assume that $\phi = 1$, so that $p_i^*(t) = m(t)$; thus $x(t) = (1/T) \int_{\tau=0}^T E_t m(t + \tau) d\tau$.

- (a) Suppose that initially $m(t) = gt$ ($g > 0$), and that $E_t m(t + \tau)$ is therefore $(t + \tau)g$. What are $x(t)$, $p(t)$, and $y(t) = m(t) - p(t)$?
- (b) Suppose that at time zero the government announces that it is steadily reducing money growth to zero over the next interval T of time. Thus $m(t) = t[1 - (t/2T)]g$ for $0 < t < T$, and $m(t) = gT/2$ for $t \geq T$. The change is unexpected, so that prices set before $t = 0$ are as in part (a).
 - (i) Show that if $x(t) = gT/2$ for all $t > 0$, then $p(t) = m(t)$ for all $t > 0$, and thus that output is the same as it would be without the change in policy.
 - (ii) For $0 < t < T$, are the prices that firms set more than, less than, or equal to $gT/2$? What about for $T \leq t \leq 2T$? Given this, how does output during the period $(0, 2T)$ compare with what it would be without the change in policy?

6.12. State-dependent pricing with both positive and negative inflation. (This follows Caplin and Leahy, 1991.) Consider an economy like that of the Caplin-Spulber model. Suppose, however, that m can either rise or fall, and that firms therefore follow a two-sided S policy: if $p_i - p_i^*(t)$ reaches either S or $-S$, firm i changes p_i so that $p_i - p_i^*(t)$ equals zero. As in the Caplin-Spulber model, changes in m are continuous.

Assume for simplicity that $p_i^*(t) = m(t)$. In addition, assume that $p_i - p_i^*(t)$ is initially distributed uniformly over some interval of width S ; that is, $p_i - p_i^*(t)$ is distributed uniformly on $[X, X + S]$ for some X between $-S$ and 0 . This is shown in Figure 6.10: the distribution of $p_i - p_i^*(t)$ is an "elevator" of height S in a "shaft" of height $2S$.

- (a) Explain why, given these assumptions, $p_i - p_i^*(t)$ continues to be distributed uniformly over some interval of width S . (In terms of the diagram, this means that although the elevator may move in the shaft, it remains of height S .)
- (b) Are there any positions of the elevator (that is, any values of X) where an infinitesimal increase in m of dm raises average prices by less than dm ? by more than dm ? by exactly dm ? Thus, what does this model imply about the real effects of monetary shocks?

6.13. Consider an economy consisting of some firms with flexible prices and some with rigid prices. Let p^f denote the price set by a representative flexible-price firm and p^r the price set by a representative rigid-price firm. Flexible-price firms set their prices after m is known; rigid-price firms set their prices before m is known. Thus flexible-price firms set $p^f = p_i^* = (1 - \phi)p + \phi m$, and rigid-price firms set $p^r = E p_i^* = (1 - \phi)E p + \phi E m$, where E denotes the expectation of a variable as of when the rigid-price firms set their prices.

Assume that fraction q of firms have rigid prices, so that $p = qp^r + (1 - q)p^f$.

- (a) Find p^f in terms of p^r , m , and the parameters of the model (ϕ and q).
- (b) Find p^r in terms of $E m$ and the parameters of the model.

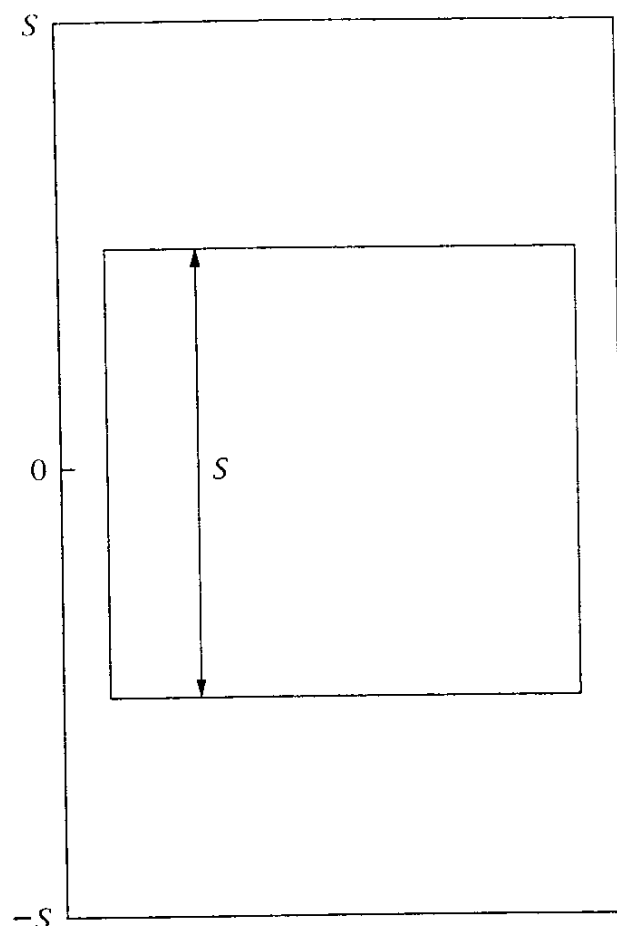


FIGURE 6.10 The distribution of $p_i - p_i^*(t)$ in the Caplin-Leahy model

- (c) (i) Do anticipated changes in m (that is, changes that are expected as of when rigid-price firms set their prices) affect y ? Why or why not?
- (ii) Do unanticipated changes in m affect y ? Why or why not?

6.14. Consider an economy consisting of many imperfectly competitive, price-setting firms. The profits of the representative firm, firm i , depend on aggregate output, y , and the firm's real price, r_i : $\pi_i = \pi(y, r_i)$, where $\pi_{22} < 0$ (subscripts denote partial derivatives). Let $r^*(y)$ denote the profit-maximizing price as a function of y ; note that $r^*(y)$ is characterized by $\pi_2(y, r^*(y)) = 0$.

Assume that output is at some level y_0 , and that firm i 's real price is $r^*(y_0)$. Now suppose there is a change in the money supply, and suppose that other firms do not change their prices and that aggregate output therefore changes to some new level, y_1 .

- (a) Explain why firm i 's incentive to adjust its price is given by $G = \pi(y_1, r^*(y_1)) - \pi(y_1, r^*(y_0))$.
- (b) Use a second-order Taylor approximation of this expression in y_1 around $y_1 = y_0$ to show that $G \approx -\pi_{22}(y_0, r^*(y_0))[r^*(y_0)]^2(y_1 - y_0)^2/2$.
- (c) What component of this expression corresponds to the degree of real rigidity? What component corresponds to the degree of insensitivity of the profit function?

6.15. Multiple equilibria with menu costs. (This follows Ball and D. Romer, 1991.)

Consider an economy consisting of many imperfectly competitive firms. The profits that a firm loses relative to what it obtains with $p_i = p^*$ are $K(p_i - p^*)^2$, $K > 0$. As usual, $p^* = p + \phi y$ and $y = m - p$. Each firm faces a fixed cost Z of changing its nominal price.

Initially m is zero and the economy is at its flexible-price equilibrium, which is $y = 0$ and $p = m = 0$. Now suppose m changes to m' .

- Suppose that fraction f of firms change their prices. Since the firms that change their prices charge p^* and the firms that do not charge zero, this implies $p = fp^*$. Use this fact to find p , y , and p^* as functions of m' and f .
- Plot a firm's incentive to adjust its price, $K(0 - p^*)^2 = Kp^{*2}$, as a function of f . Be sure to distinguish the cases $\phi < 1$ and $\phi > 1$.
- A firm adjusts its price if the benefit exceeds Z , does not adjust if the benefit is less than Z , and is indifferent if the benefit is exactly Z . Given this, can there be a situation where both adjustment by all firms and adjustment by no firms are equilibria? Can there be a situation where neither adjustment by all firms nor adjustment by no firms are equilibria?

6.16. (This follows Diamond, 1982.)⁴¹ Consider an island consisting of N people and many palm trees. Each person is in one of two states, not carrying a coconut and looking for palm trees (state P) or carrying a coconut and looking for other people with coconuts (state C). If a person without a coconut finds a palm tree, he or she can climb the tree and pick a coconut; this has a cost (in utility units) of c . If a person with a coconut meets another person with a coconut, they trade and eat each other's coconuts; this yields $\bar{u}$ units of utility for each of them. (People cannot eat coconuts that they have picked themselves.)

A person looking for coconuts finds palm trees at rate b per unit time. A person carrying a coconut finds trading partners at rate aL per unit time, where L is the total number of people carrying coconuts. a and b are exogenous.

Individuals' discount rate is r . Focus on steady states; that is, assume that L is constant.

- Explain why, if everyone in state P climbs a palm tree whenever he or she finds one, then $rV_P = b(V_C - V_P - c)$, where V_P and V_C are the values of being in the two states.
- Find the analogous expression for V_C .
- Solve for $V_C - V_P$, V_C , and V_P in terms of r , b , c , $\bar{u}$, a , and L .
- What is L , still assuming that anyone in state P climbs a palm tree whenever he or she finds one? Assume for simplicity that $aN = 2b$.

⁴¹The solution to this problem requires dynamic programming (see Section 10.4).

- (e) For what values of c is it a steady-state equilibrium for anyone in state P to climb a palm tree whenever he or she finds one? (Continue to assume $aN = 2b$.)
- (f) For what values of c is it a steady-state equilibrium for no one who finds a tree to climb it? Are there values of c for which there is more than one steady-state equilibrium? If there are multiple equilibria, does one involve higher welfare than the other? Explain intuitively.

Chapter 7

CONSUMPTION

This chapter and the next investigate households' consumption choices and firms' investment decisions in more detail. Consumption and investment are important to both growth and fluctuations. With regard to growth, the division of society's resources between current consumption and various types of investment—in physical capital, human capital, and research and development—is central to standards of living in the long run. That division is determined by the interaction of households' allocation of their incomes between consumption and saving given the rates of return and other constraints they face, and firms' investment demand given the interest rates and other constraints they face. With regard to fluctuations, consumption and investment make up the vast majority of the demand for goods. Thus if we wish to understand how such forces as government purchases, technology, and monetary policy affect aggregate output, we must understand how consumption and investment are determined.

There are two other reasons for studying consumption and investment. First, they introduce some important issues involving financial markets. Financial markets affect the macroeconomy mainly through their impact on consumption and investment. In addition, consumption and investment have important feedback effects on financial markets. We will investigate the interaction between financial markets and consumption and investment both in cases where financial markets function perfectly and in cases where they do not.

Second, much of the most insightful empirical work in macroeconomics over the past twenty years has been concerned with consumption and investment. These two chapters therefore have an unusually intensive empirical focus.

approximately tracks their income, but they have a small amount of saving that they use in the event of sharp falls in income or emergency spending needs. In the terminology of Deaton (1991), most households exhibit *buffer-stock* saving behavior. As a result, a small fraction of households hold the vast majority of wealth.

At least three explanations of buffer-stock saving have been proposed. First, Shefrin and Thaler (1988) argue that consumption behavior is not well described by complete intertemporal optimization (see also Laibson, 1993). Instead, individuals have a set of rules of thumb that they use to guide their consumption behavior. Examples of these rules of thumb are that it is usually reasonable to spend one's current income, but that assets should be dipped into only in exceptional circumstances. Such rules of thumb may lead consumers to use saving and borrowing to smooth short-run income fluctuations, and thus cause consumption to follow the predictions of the permanent-income hypothesis reasonably well at short horizons. But they may also cause consumption to track income fairly closely over long horizons.

Second, Deaton (1991) and Carroll (1992) argue that buffer-stock saving arises from a combination of a high discount rate, a precautionary-saving motive, and some reason that households do not go heavily into debt. In Deaton's analysis, the reason for the absence of debt is the presence of liquidity constraints. In Carroll's, it is that the marginal utility of consumption approaches infinity as consumption becomes sufficiently low; as a result, households are unwilling to risk the very low consumption that would occur if they were in debt and their future income turned out to be low. The combination of the high discount rate and the inability or unwillingness to go into debt causes households' wealth to be approximately zero, and thus causes consumption to approximately track income. But even with a relatively high discount rate, the positive third derivative of the utility function causes households to view the risks of sharp falls in consumption and sharp rises as asymmetric; as a result, they typically keep a small amount of savings to use in the event of large falls in income.

Third, Hubbard, Skinner, and Zeldes (1994a, 1994b) suggest an explanation of buffer-stock saving that is close in spirit to the permanent-income hypothesis. The key elements of their explanation, aside from intertemporal optimization, are a precautionary-saving motive and the fact that welfare programs provide insurance against very low levels of consumption. For households that face a nonnegligible probability of going on welfare, the presence of welfare discourages saving in two ways: it directly provides insurance against unfavorable realizations of income, and it imposes extremely high implicit tax rates on asset holdings. Nonetheless, the precautionary-saving motive causes these households to typically hold some assets when their consumption is above the guaranteed floor. For households whose income prospects are favorable enough that the possibility of going on welfare is negligible, on the other hand, consumption is determined by conventional intertemporal optimization; thus they ex-

hibit conventional life-cycle saving. Hubbard, Skinner, and Zeldes therefore argue that the different patterns of wealth accumulation of the poor and the rich can be explained without appealing to differences in their preferences.

Problems

- 7.1. The average income of farmers is less than the average income of non-farmers, but fluctuates more from year to year. Given this, how does the permanent-income hypothesis predict that estimated consumption functions for farmers and nonfarmers differ?
- 7.2. **The time-averaging problem.** (Working, 1960.) Actual data do not give consumption at a point in time, but average consumption over an extended period, such as a quarter. This problem asks you to examine the effects of this fact.
- Suppose that consumption follows a random walk: $C_t = C_{t-1} + e_t$, where e is white noise. Suppose, however, that the data provide average consumption over two-period intervals; that is, one observes $(C_t + C_{t+1})/2$, $(C_{t+2} + C_{t+3})/2$, and so on.
- Find an expression for the change in measured consumption from one two-period interval to the next in terms of the e 's.
 - Is the change in measured consumption uncorrelated with the previous value of the change in measured consumption? In light of this, is measured consumption a random walk?
 - Given your result in part (a), is the change in consumption from one two-period interval to the next necessarily uncorrelated with anything known as of the first of these two-period intervals? Is it necessarily uncorrelated with anything known as of the two-period interval immediately preceding the first of the two-period intervals?
 - Suppose that measured consumption for a two-period interval is not the average over the interval, but consumption in the second of the two periods. That is, one observes C_{t+1} , C_{t+3} , and so on. In this case, is measured consumption a random walk?
- 7.3. (This follows Hansen and Singleton, 1983.) Suppose instantaneous utility is of the constant-relative-risk-aversion form, $u(C_t) = C_t^{1-\theta}/(1-\theta)$, $\theta > 0$. Assume that the real interest rate, r , is constant but not necessarily equal to the discount rate, ρ .
- Find the Euler equation relating C_t to expectations concerning C_{t+1} .
 - Suppose that the log of income is distributed normally, and that as a result the log of C_{t+1} is distributed normally; let σ^2 denote its variance conditional on information available at time t . Rewrite the expression in part (a) in terms of $\ln C_t$, $E_t[\ln C_{t+1}]$, σ^2 , and the parameters r , ρ , and θ . (Hint: if a variable x is distributed normally with mean μ and variance V , $E[e^x] = e^{\mu + V/2}$.)

- (c) Show that if r and σ^2 are constant over time, the result in part (b) implies that the log of consumption follows a random walk with drift: $\ln C_{t+1} = a + \ln C_t + u_{t+1}$, where u is white noise.
- (d) How do changes in each of r and σ^2 affect expected consumption growth, $E_t[\ln C_{t+1} - \ln C_t]$? Interpret the effect of σ^2 on expected consumption growth in light of the discussion of precautionary saving in Section 7.6.

7.4. A framework for investigating excess smoothness. Suppose that C_t equals $[r/(1+r)][A_t + \sum_{s=0}^{\infty} E_t\{Y_{t-s}\}/(1+r)^s]$, and that $A_{t+1} = (1+r)(A_t + Y_t - C_t)$.

- (a) Show that these assumptions imply that $E_t[C_{t+1}] = C_t$ (and thus that consumption follows a random walk) and that $\sum_{s=0}^{\infty} E_t[C_{t-s}]/(1+r)^s = A_t + \sum_{s=0}^{\infty} E_t\{Y_{t-s}\}/(1+r)^s$.
- (b) Suppose that $\Delta Y_t = \phi \Delta Y_{t-1} + u_t$, where u is white noise. Suppose that Y_t exceeds $E_{t-1}[Y_t]$ by one unit (that is, suppose $u_t = 1$). By how much does consumption increase?
- (c) For the case of $\phi > 0$, which has a larger variance, the innovation in income, u_t , or the innovation in consumption, $C_t - E_{t-1}[C_t]$? Do consumers use saving and borrowing to smooth the path of consumption relative to income in this model? Explain.

7.5. Consider the two-period setup analyzed in Section 7.4. Suppose that the government initially raises revenue only by taxing interest income. Thus the individual's budget constraint is $C_1 + C_2/[1 + (1-\tau)r] \leq Y_1 + Y_2/[1 + (1-\tau)r]$, where τ is the tax rate. The government's revenue is zero in period 1 and $\tau r(Y_1 - C_1^0)$ in period 2, where C_1^0 is the individual's choice of C_1 given this tax rate. Now suppose the government eliminates the taxation of interest income and instead institutes lump-sum taxes of amounts T_1 and T_2 in the two periods; thus the individual's budget constraint is now $C_1 + C_2/(1+r) \leq (Y_1 - T_1) + (Y_2 - T_2)/(1+r)$. Assume that Y_1 , Y_2 , and r are exogenous.

- (a) What condition must the new taxes satisfy so that the change does not affect the present value of government revenues?
- (b) If the new taxes satisfy the condition in part (a), is the old consumption bundle, (C_1^0, C_2^0) , not affordable, just affordable, or affordable with room to spare?
- (c) If the new taxes satisfy the condition in part (a), does first-period consumption rise, fall, or stay the same?

7.6. Consumption of durable goods. (Mankiw, 1982.) Suppose that, as in Section 7.2, the instantaneous utility function is quadratic and the interest rate and the discount rate are zero. Suppose, however, that goods are durable; specifically, $C_t = (1-\delta)C_{t-1} + E_t$, where E_t is purchases in period t and $0 \leq \delta < 1$.

- (a) Consider a marginal reduction in purchases in period t of dE_t . Find values of dE_{t+1} and dE_{t+2} such that the combined changes in E_t , E_{t+1} , and E_{t+2} leave the present value of spending unchanged (so $dE_t + dE_{t+1} + dE_{t+2} = 0$) and leave C_{t+2} unchanged (so $(1-\delta)^2 dE_t + (1-\delta)dE_{t+1} + dE_{t+2} = 0$).
- (b) What is the effect of the change in part (a) on C_t and C_{t+1} ? What is the effect on expected utility?

- (c) What condition must C_t and $E_t[C_{t+1}]$ satisfy for the change in part (a) not to affect expected utility? Does C follow a random walk?
- (d) Does E follow a random walk? (Hint: write $E_t - E_{t-1}$ in terms of $C_t - C_{t-1}$ and $C_{t-1} - C_{t-2}$.) Explain intuitively. If $\delta = 0$, what is the behavior of E ?
- 7.7. Consider a stock that pays dividends of D_t in period t and whose price in period t is P_t . Assume that consumers are risk-neutral and have a discount rate of r ; thus they maximize $E[\sum_{t=0}^{\infty} C_t / (1+r)^t]$.
- (a) Show that equilibrium requires $P_t = E_t[(D_{t+1} + P_{t+1}) / (1+r)]$ (assume that if the stock is sold, this happens after that period's dividends have been paid).
- (b) Assume that $\lim_{s \rightarrow \infty} E_t[P_{t+s} / (1+r)^s] = 0$ (this is a *no-bubbles* condition; see the next problem). Iterate the expression in part (a) forward to derive an expression for P_t in terms of expectations of future dividends.
- 7.8. **Bubbles.** Consider the setup of the previous problem without the assumption that $\lim_{s \rightarrow \infty} E_t[P_{t+s} / (1+r)^s] = 0$.
- (a) **Deterministic bubbles.** Suppose that P_t equals the expression derived in part (b) of Problem 7.7 plus $(1+r)^t b$, $b > 0$.
- (i) Is consumers' first-order condition derived in part (a) of Problem 7.7 still satisfied?
- (ii) Can b be negative? (Hint: consider the strategy of never selling the stock.)
- (b) **Bursting bubbles.** (Blanchard, 1979.) Suppose that P_t equals the expression derived in part (b) of Problem 7.7 plus q_t , where q_t equals $(1+r)q_{t-1}/\alpha$ with probability α and equals zero with probability $1 - \alpha$.
- (i) Is consumers' first-order condition derived in part (a) of Problem 7.7 still satisfied?
- (ii) If there is a bubble at time t (that is, if $q_t > 0$), what is the probability that the bubble has burst by time $t+s$ (that is, that $q_{t+s} = 0$)? What is the limit of this probability as s approaches infinity?
- (c) **Intrinsic bubbles.** (Froot and Obstfeld, 1991.) Suppose that dividends follow a random walk: $D_t = D_{t-1} + e_t$, where e is white noise.
- (i) In the absence of bubbles, what is the price of the stock in period t ?
- (ii) Suppose that P_t equals the expression derived in (i) plus b_t , where $b_t = (1+r)b_{t-1} + ce_t$, $c > 0$. Is consumers' first-order condition derived in part (a) of Problem 7.7 still satisfied? In what sense do stock prices overreact to changes in dividends?
- 7.9. **The Lucas asset-pricing model.** (Lucas, 1978.) Suppose the only assets in the economy are infinitely-lived trees. Output equals the fruit of the trees, which is exogenous and cannot be stored; thus $C_t = Y_t$, where Y_t is the exogenously determined output per person and C_t is consumption per person. Assume that initially each consumer owns the same number of trees. Since all consumers are assumed to be the same, this means that, in equilibrium, the behavior of the price of trees must be such that, each period, the representative

consumer does not want to either increase or decrease his or her holdings of trees.

Let P_t denote the price of a tree in period t (assume that if the tree is sold, the sale occurs after the existing owner receives that period's output). Finally, assume that the representative consumer maximizes $E[\sum_{t=0}^{\infty} \ln C_t / (1 + \rho)^t]$.

- (a) Suppose the representative consumer reduces his or her consumption in period t by an infinitesimal amount, uses the resulting saving to increase his or her holdings of trees, and then sells these additional holdings in period $t + 1$. Find the condition that C_t and expectations involving Y_{t+1} , P_{t+1} , and C_{t+1} must satisfy for this change not to affect expected utility. Solve this condition for P_t in terms of Y_t and expectations involving Y_{t+1} , P_{t+1} , and C_{t+1} .
- (b) Assume that $\lim_{s \rightarrow \infty} E_t[(P_{t+s}/Y_{t+s})/(1 + \rho)^s] = 0$. Given this assumption, iterate your answer to part (a) forward to solve for P_t . (Hint: use the fact that $C_{t+s} = Y_{t+s}$ for all s .)
- (c) Explain intuitively why an increase in expectations of future dividends does not affect the price of the asset.
- (d) Does consumption follow a random walk in this model?

7.10. The equity premium and the concentration of aggregate shocks. (Mankiw, 1986b.) Consider an economy with two possible states, each of which occurs with probability $\frac{1}{2}$. In the good state, each individual's consumption is 1. In the bad state, fraction λ of the population consumes $1 - (\phi/\lambda)$ and the remainder consumes 1, where $0 < \phi < 1$ and $\phi \leq \lambda \leq 1$. ϕ measures the reduction in average consumption in the bad state, and λ measures how broadly that reduction is shared.

Consider two assets, one that pays off 1 unit in the good state and one that pays off 1 unit in the bad state. Let p denote the relative price of the bad-state asset to the good-state asset.

- (a) Consider an individual whose initial holdings of the two assets are zero, and consider the experiment of the individual marginally reducing (that is, selling short) his or her holdings of the good-state asset and using the proceeds to purchase more of the bad-state asset. Derive the condition for this change not to affect the individual's expected utility.
- (b) Since consumption in the two states is exogenous and individuals are ex ante identical, p must adjust to the point where it is an equilibrium for individuals' holdings of both assets to be zero. Solve the condition derived in part (a) for this equilibrium value of p in terms of ϕ , λ , $U'(1)$, and $U'(1 - (\phi/\lambda))$.
- (c) Find $\partial p / \partial \lambda$.
- (d) Show that if utility is quadratic, $\partial p / \partial \lambda = 0$.
- (e) Show that if $U'''(\cdot)$ is everywhere positive, $\partial p / \partial \lambda < 0$.

Chapter 8

INVESTMENT

This chapter investigates the demand for investment. As described at the beginning of Chapter 7, there are two main reasons for studying investment. First, the combination of firms' investment demand and households' saving supply determines how much of an economy's output is invested; as a result, investment demand is potentially important to the behavior of standards of living over the long run. Second, investment is highly volatile; thus investment demand may be important to short-run fluctuations.

Section 8.1 presents a baseline model of investment where firms face a perfectly elastic supply of capital goods and can adjust their capital stocks costlessly. We will see that this model, even though it is a natural one to consider, provides little insight into actual investment. It implies, for example, that discrete changes in the economic environment (such as discrete changes in interest rates) produce infinite rates of investment or disinvestment.

Sections 8.2 through 8.5 therefore develop and analyze the *q theory* model of investment. The model's key assumption is that firms face costs of adjusting their capital stocks. As a result, the model avoids the unreasonable implications of the baseline case and provides a useful framework for analyzing the effects that expectations and current conditions have on investment.

Sections 8.6 and 8.7 introduce two important extensions of the model: Section 8.6 considers uncertainty, and Section 8.7 investigates financial-market imperfections. Finally, Section 8.8 presents some empirical tests and applications of the models.

8.1 Investment and the Cost of Capital

The Desired Capital Stock

Consider a firm that can rent capital at a price of r_K . The firm's profits at a point in time are given by $\pi(K, X_1, X_2, \dots, X_n) - r_K K$, where K is the amount of capital the firm rents and the X 's are variables that it takes as given. In the

These papers' findings are representative of the findings in this literature: the evidence consistently suggests that financial-market imperfections are important to investment. Precisely what form those imperfections take, and how important they are quantitatively, remain open questions.

Problems

- 8.1. Consider a firm that produces output using a Cobb–Douglas combination of capital and labor: $Y = K^\alpha L^{1-\alpha}$, $0 < \alpha < 1$. Suppose that the firm's price is fixed in the short run; thus it takes both the price of its product, P , and the quantity, Y , as given. Finally, input markets are competitive; thus the firm takes the wage, W , and the rental price of capital, r_K , as given.
- What is the firm's choice of L given P , Y , W , and K ?
 - Given this choice of L , what are profits as a function of P , Y , W , and K ?
 - Find the first-order condition for the profit-maximizing choice of K . Is the second-order condition satisfied?
 - Solve the first-order condition in part (c) for K as a function of P , Y , W , and r_K . How, if at all, do changes in each of these variables affect K ?
- 8.2. Corporations in the United States are allowed to subtract depreciation allowances from their taxable income. The depreciation allowances are based on the purchase price of the capital; a corporation that buys a new capital good at time t can deduct fraction $D(s)$ of the purchase price from its taxable income at time $t + s$. Depreciation allowances often take the form of *straight-line depreciation*: $D(s)$ equals $1/T$ for $s \in [0, T]$, and equals zero for $s > T$, where T is the *tax life* of the capital good.
- Assume straight-line depreciation. If the marginal corporate income tax rate is constant at τ and the interest rate is constant at i , by how much does purchasing a unit of capital at a price of P_K reduce the present value of the firm's corporate tax liabilities as a function of T , τ , i , and P_K ? Thus, what is the after-tax price of the capital good to the firm?
 - Suppose that $i = r + \pi$, and that π increases with no change in r . How does this affect the after-tax price of the capital good to the firm?
- 8.3. The major feature of the tax code that affects the user cost of capital in the case of owner-occupied housing in the United States is that nominal interest payments are tax deductible. Thus the after-tax real interest rate relevant to home ownership is $r - \pi i$, where r is the pretax real interest rate, i is the nominal interest rate, and τ is the marginal tax rate. In this case, how does an increase in inflation for a given r affect the user cost of capital and the desired capital stock?
- 8.4. **Using the calculus of variations to solve the social planner's problem in the Ramsey model.** Consider the social planner's problem that we analyzed in Section 2.4: the planner wants to maximize $\int_{t=0}^{\infty} e^{-\beta t} [c(t)^{1-\theta}/(1-\theta)] dt$ subject to $\dot{k}(t) = f(k(t)) - c(t) - (n + g)k(t)$.

- (a) What is the current-value Hamiltonian? What variables are the control variable, the state variable, and the costate variable?
- (b) Find the three conditions that characterize optimal behavior analogous to equations (8.18), (8.19), and (8.20), in Section 8.2.
- (c) Show that the first two conditions in part (b), together with the fact that $f'(k(t)) = r(t)$, imply the Euler equation (equation [2.19]).
- (d) Let μ denote the costate variable. Show that $[\dot{\mu}(t)/\mu(t)] - \beta = (n + g) - r(t)$, and thus that $e^{-\beta t} \mu(t)$ is proportional to $e^{-R(t)} e^{(n+g)t}$. Show that this implies that the transversality condition in part (b) holds if and only if the budget constraint, equation (2.12), holds with equality.
- 8.5. Consider the model of investment in Sections 8.2-8.5. Describe the effects of each of the following changes on the $\dot{K} = 0$ and $\dot{q} = 0$ loci, on K and q at the time of the change, and on their behavior over time. In each case, assume that K and q are initially at their long-run equilibrium values.
- (a) A war destroys half of the capital stock.
- (b) The government taxes returns from owning firms at rate τ .
- (c) The government taxes investment. Specifically, firms pay the government γ for each unit of capital they acquire, and receive a subsidy of γ for each unit of disinvestment.
- 8.6. Consider the model of investment in Sections 8.2-8.5. Suppose it becomes known at some date that there will be a one-time capital levy; specifically, capital holders will be taxed an amount equal to fraction f of the value of their capital holdings after time T . Assume the industry is initially in long-run equilibrium. What happens at the time of this news? How do K and q behave between the time of the news and the time the levy is imposed? What happens to K and q at the time of the levy? How do they behave thereafter? (Hint: is q anticipated to change discontinuously at the time of the levy?)
- 8.7. **A model of the housing market.** (This follows Poterba, 1984.) Let H denote the stock of housing, I the rate of investment, p_H the real price of housing, and R the rent. Assume that I is increasing in p_H , so that $I = I(p_H)$, with $I'(\bullet) > 0$, and that $\dot{H} = I - \delta H$. Assume also that the rent is a decreasing function of H : $R = R(H)$, $R'(\bullet) < 0$. Finally, assume that rental income plus capital gains must equal the exogenous required rate of return, r : $(R + \dot{p}_H)/p_H = r$.
- (a) Sketch the set of points in (H, p_H) space such that $\dot{H} = 0$. Sketch the set of points such that $\dot{p}_H = 0$.
- (b) What are the dynamics of H and p_H in each region of the resulting diagram? Sketch the saddle path.
- (c) Suppose the market is initially in long-run equilibrium, and that there is an unexpected permanent increase in r . What happens to H and p_H at the time of the change? How do H , p_H , I , and R behave over time following the change?
- (d) Suppose the market is initially in long-run equilibrium, and that it becomes known that there will be a permanent increase in r time T in the

future. What happens to H and p_H at the time of the news? How do H , p_H , I , and R behave between the time of the news and the time of the increase? What happens to them when the increase occurs? How do they behave after the increase?

(e) Are adjustment costs internal or external in this model? Explain.

(f) Why is the $\dot{H} = 0$ locus not horizontal in this model?

8.8. Suppose that the costs of adjustment exhibit constant returns in $\dot{\kappa}$ and κ . Specifically, suppose they are given by $C(\dot{\kappa}/\kappa)\kappa$, where $C(0) = 0$, $C'(0) = 0$, $C''(\cdot) > 0$. In addition, suppose capital depreciates at rate δ ; thus $\dot{\kappa}(t) = I(t) - \delta\kappa(t)$. Consider the representative firm's maximization problem.

(a) What is the current-value Hamiltonian?

(b) Find the three conditions that characterize optimal behavior analogous to equations (8.18), (8.19), and (8.20), in Section 8.2.

(c) Show that the condition analogous to (8.18) implies that the growth rate of each firm's capital stock, and thus the growth rate of the aggregate capital stock, is determined by q . In (K, q) space, what is the $\dot{K} = 0$ locus?

(d) Substitute your result in part (c) into the condition analogous to (8.19) to express $\dot{q}$ in terms of K and q .

(e) In (K, q) space, what is the slope of the $\dot{q} = 0$ locus at the point where $q = 1$?

8.9. Suppose that $\pi(K) = a - bK$ and $C(I) = \alpha I^2/2$.

(a) What is the $\dot{q} = 0$ locus? What is the long-run equilibrium value of K ?

(b) What is the slope of the saddle path? (Hint: use the approach in Section 2.6.)

8.10. Consider the model of investment under uncertainty with a constant interest rate in Section 8.6. Suppose that, as in Problem 8.9, $\pi(K) = a - bK$ and that $C(I) = \alpha I^2/2$; in addition, suppose that what is uncertain is future values of a . This problem asks you to show that it is an equilibrium for $q(t)$ and $K(t)$ to have the values at each point in time that they would if there were no uncertainty about the path of a . Specifically, let $\hat{q}(t + \tau, t)$ and $\hat{K}(t + \tau, t)$ be the paths q and K would take after time t if $a(t + \tau)$ were certain to equal $E_t[a(t + \tau)]$ for all $\tau \geq 0$.

(a) Show that if $E_t[q(t + \tau)] = \hat{q}(t + \tau, t)$ for all $\tau \geq 0$, then $E_t[K(t + \tau)] = \hat{K}(t + \tau, t)$ for all $\tau \geq 0$.

(b) Use equation (8.26) to show that this implies that if $E_t[q(t + \tau)] = \hat{q}(t + \tau, t)$, then $q(t) = \hat{q}(t, t)$, and thus that $\dot{K}(t) = N[\hat{q}(t, t) - 1]/\alpha$, where N is the number of firms.

8.11. (This follows Bernanke, 1983a, and Dixit and Pindyck, 1994.) Consider a firm that is contemplating undertaking an investment with a cost of I . There are two periods. The investment will pay off π_1 in period 1 and π_2 in period 2. π_1

is certain, but π_2 is uncertain. The firm maximizes expected profits and, for simplicity, the interest rate is zero.

(a) Suppose the firm's only choices are to undertake the investment in period 1 or not to undertake it at all. Under what condition will the firm undertake the investment?

(b) Suppose the firm also has the possibility of undertaking the investment in period 2, after the value of π_2 is known; in this case the investment pays off only π_2 . Is it possible for the firm's expected profits to be higher if it does not invest in period 1 than if it does even if the condition in (a) is satisfied?

(c) Define the cost of waiting as π_1 , and define the benefit of waiting as $\text{Prob}(\pi_2 < I)E[I - \pi_2 \mid \pi_2 < I]$. Explain why these represent the cost and the benefit of waiting. Show that the difference in the firm's expected profits between not investing in period 1 and investing in period 1 equals the benefit of waiting minus the cost.

8.12. The Modigliani-Miller theorem. (Modigliani and Miller, 1958.) Consider the analysis of the effects of uncertainty about discount factors in Section 8.6. Suppose, however, that the firm finances its investment using a mix of equity and risk-free debt. Specifically, consider the financing of the marginal unit of capital. The firm issues quantity b of bonds; each bond pays one unit of output with certainty at time $t + \tau$ for all $\tau \geq 0$. Equity holders are the residual claimant; thus they receive $\pi(K(t + \tau)) - b$ at $t + \tau$ for all $\tau \geq 0$.

(a) Let $P(t)$ denote the value of a unit of debt at t , and $V(t)$ the value of the equity in the marginal unit of capital. Find expressions analogous to (8.29) for $P(t)$ and $V(t)$.

(b) How, if at all, does the division of financing between bonds and equity affect the market value of the claims on the unit of capital, $P(t)b + V(t)$? Explain intuitively.

(c) More generally, suppose the firm finances the investment by issuing n financial instruments. Let $d_i(t + \tau)$ denote the payoff to instrument i at time $t + \tau$; the payoffs satisfy $d_1(t + \tau) + \cdots + d_n(t + \tau) = \pi(K(t + \tau))$, but are otherwise unrestricted. How, if at all, does the total value of the n assets depend on how the total payoff is divided among the assets?

(d) Return to the case of debt and equity finance. Suppose, however, that the firm's profits are taxed at rate θ , and that interest payments are tax deductible. Thus the payoff to bond holders is the same as before, but the payoff to equity holders at time $t + \tau$ is now $(1 - \theta)[\pi(K(t + \tau)) - b]$. Does the result in part (b) still hold? Explain.

ones.⁴² And finally, highly variable inflation (or even higher average inflation alone) can also discourage long-term investment because firms and individuals view it as a symptom of a government that is functioning badly, and that may therefore resort to confiscatory taxation or other policies that are highly detrimental to capital-holders.

Empirically, there is a strong negative association between inflation and investment, and between inflation and growth (Fischer, 1991, 1993; Rudebusch and Wilcox, 1994). At this point, however, there is little evidence concerning whether these relationships are causal. It is not difficult to think of reasons that the associations might not represent true effects of inflation. In the short run, negative supply shocks are associated with both higher inflation and lower productivity growth. In the long run, governments that follow policies detrimental to growth—protectionism, large budget deficits, and so on—are likely to also pursue policies that result in inflation (Sala-i-Martin, 1991b).⁴³

For high inflation rates, one can argue that the issue of whether the association between inflation and growth represents an effect of inflation on growth is of limited relevance. For a country to reduce inflation from very high levels, it is likely to need to adopt a broad range of budgetary and policy reforms. Thus growth is likely to rise, even though it may be the other reforms and not the reductions in inflation that bring it about.⁴⁴ In contrast, inflation can be reduced from moderate to low levels without fundamental policy reforms. Thus for moderate and low inflation, the issue of causation is crucial.

Potential Benefits of Inflation

So far we have considered only costs of inflation. But inflation can have benefits as well. Tobin (1972) observes that if it is particularly difficult for firms to cut nominal wages, real wages can make needed adjustments to sector-specific shocks more rapidly when inflation is higher. Summers (1991) notes that since nominal interest rates cannot be negative, low inflation (by causing usual nominal rates to be low) may limit the Federal Reserve's ability to stimulate aggregate demand in response to contractionary shocks. And just as inflation above some level can disrupt long-term planning and increase uncertainty, so too can inflation below some level. Given that average inflation has been significantly positive over the last several decades, it is not

⁴²If these costs of inflation variability are large, however, there may be large incentives for individuals and firms to write contracts in real rather than nominal terms, or to create markets that allow them to insure against inflation risk. Thus a complete account of large costs of inflation through these channels must explain the absence of these institutions.

⁴³Moreover, the estimates of Fischer and of Rudebusch and Wilcox suggest that at moderate inflation rates, a 1-percentage-point reduction in inflation is associated with roughly a 0.2-percentage-point increase in growth. This association is so large that it is difficult to identify plausible ways that it can represent an effect of inflation on growth.

⁴⁴This argument is due to Allan Meltzer.

clear that zero inflation minimizes uncertainty and is least disruptive. Finally, as described above, inflation is a potential source of revenue for the government; under some conditions it is optimal for the government to use this revenue source in addition to more conventional taxes.

In addition, it is possible that the public's aversion to inflation represents not some deep understanding of the costs of inflation that has eluded economists, but a misapprehension. For example, Katona (1976) argues that the public perceives how inflation affects prices but not wages. Thus when inflation rises, individuals attribute only the faster growth of prices to the increase; they therefore incorrectly conclude that it has reduced their standard of living. Alternatively, individuals may dislike inflation just because times of high inflation are also times of low real growth; but if the high inflation is not in fact the source of the low growth, again inflation does not actually make them worse off.

Concluding Comments

As this discussion shows, research has not yet yielded any firm conclusions about the costs of inflation and the optimal rate of inflation. Thus economists and policymakers must rely on their judgment in weighing the different considerations. Loosely speaking, they fall into two groups. One group views inflation as pernicious, and believes that policy should focus on eliminating inflation and pay virtually no attention to other considerations. Members of this group generally believe that policy should aim for zero inflation or moderate deflation. The other group concludes that extremely low inflation is of little benefit, or perhaps even harmful, and believes that policy should aim to keep average inflation low to moderate but should keep other objectives in mind. The opinions of members of this group about the level of inflation that policy should aim for generally range from a few percent to close to 10 percent.

Problems

9.1. Consider a discrete-time version of the analysis of money growth, inflation, and real balances in Section 9.2. Suppose that money demand is given by $m_t - p_t = c - b(E_t p_{t+1} - p_t)$, where m and p are the logs of the money stock and the price level, and where we are implicitly assuming that output and the real interest rate are constant (see [9.36]).

- (a) Solve for p_t in terms of m_t and $E_t p_{t+1}$.
- (b) Use the law of iterated projections to express $E_t p_{t+1}$ in terms of $E_t m_{t+1}$ and $E_t p_{t+2}$.
- (c) Iterate this process forward to express p_t in terms of m_t , $E_t m_{t+1}$, $E_t m_{t+2}$, ... (Assume that $\lim_{i \rightarrow \infty} E_t \{ [b/(1+b)]^i p_{t+i} \} = 0$. This is a no-bubbles condition analogous to the one in Problem 7.7.)

- (d) Explain intuitively why an increase in $E_t m_{t+i}$ for any $i > 0$ raises p_t .
- (e) Suppose expected money growth is constant, so $E_t m_{t+i} = m_t + gi$. Solve for p_t in terms of m_t and g . How does an increase in g affect p_t ?

9.2. Consider a discrete-time model where prices are completely unresponsive to unanticipated monetary shocks for one period and completely flexible thereafter. Suppose the IS and LM curves are $y = c - ar$ and $m - p = b + hy - ki$, where y , m , and p are the logs of output, the money supply, and the price level; r is the real interest rate; i is the nominal interest rate; and a , h , and k are positive parameters.

Assume that initially m is constant at some level, which we normalize to zero, and that y is constant at its flexible-price level, which we also normalize to zero. Now suppose that in some period—period 1 for simplicity—the monetary authority shifts unexpectedly to a policy of increasing m by some amount $g > 0$ each period.

- (a) What are r , π^e , i , and p before the change in policy?
- (b) Once prices have fully adjusted, $\pi^e = g$. Use this fact to find r , i , and p in period 2.
- (c) In period 1, what are i , r , p , and the expectation of inflation from period 1 to period 2, $E_1[p_2] - p_1$?
- (d) What determines whether the short-run effect of the monetary expansion is to raise or lower the nominal interest rate?

9.3. Assume, as in Problem 9.2, that prices are completely unresponsive to unanticipated monetary shocks for one period and completely flexible thereafter. Assume also that $y = c - ar$ and $m - p = b + hy - ki$ hold each period. Suppose, however, that the money supply follows a random walk: $m_t = m_{t-1} + u_t$, where u_t is a mean-zero, serially uncorrelated disturbance.

- (a) Let E_t denote expectations as of period t . Explain why, for any t , $E_t[E_{t+1}[p_{t+2}] - p_{t+1}] = 0$, and thus why $E_t m_{t+1} - E_t p_{t+1} = b + h\bar{y} - k\bar{r}$.
- (b) Use the result in part (a) to solve for y_t , p_t , i_t , and r_t in terms of m_{t-1} and u_t .
- (c) Does the Fisher effect hold in this economy? That is, are changes in expected inflation reflected one-for-one in the nominal interest rate?

9.4. Suppose you want to test the hypothesis that the real interest rate is constant, so that all changes in the nominal interest rate reflect changes in expected inflation. Thus your hypothesis is $i_t = r + E_t \pi_{t+1}$.

- (a) Consider a regression of i_t on a constant and π_{t-1} . Does the hypothesis that the real interest rate is constant make a general prediction about the coefficient on π_{t-1} ? Explain. (Hint: for a univariate OLS regression, the coefficient on the right-hand-side variable equals the covariance between the right-hand-side and left-hand-side variables divided by the variance of the right-hand-side variable.)
- (b) Consider a regression of π_{t+1} on a constant and i_t . Does the hypothesis that the real interest rate is constant make a general prediction about the coefficient on i_t ? Explain.

- (c) Some argue that the hypothesis that the real interest rate is constant implies that nominal interest rates move one-for-one with actual inflation in the long run—that is, that the hypothesis implies that in a regression of i on a constant and the current and many lagged values of π , the sum of the coefficients on the inflation variables will be 1. Is this claim correct? (Hint: Suppose that the behavior of actual inflation is given by $\pi_t = \rho\pi_{t-1} + e_t$, where e is white noise.)

9.5. Policy rules, rational expectations, and regime changes. (See Lucas, 1976, and Sargent, 1983.) Suppose that aggregate supply is given by the Lucas supply curve, $y_t = \bar{y} + b(\pi_t - \pi_t^e)$, $b > 0$, and suppose that monetary policy is determined by $m_t = m_{t-1} + a + \varepsilon_t$, where ε is a white-noise disturbance. Assume that private agents do not know the current values of m_t or ε_t ; thus π_t^e is the expectation of $p_t - p_{t-1}$ given $m_{t-1}, \varepsilon_{t-1}, y_{t-1}$, and p_{t-1} . Finally, assume that aggregate demand is given by $y_t = m_t - p_t$.

- (a) Find y_t in terms of m_{t-1} , m_t , and any other variables or parameters that are relevant.
- (b) Are m_{t-1} and m_t all one needs to know about monetary policy to find y_t ? Explain intuitively.
- (c) Suppose that monetary policy is initially determined as above, with $a > 0$, and that the monetary authority then announces that it is switching to a new regime where a is zero. Suppose that private agents believe that the probability that the announcement is true is ρ . What is y_t in terms of m_{t-1} , m_t , ρ , $\bar{y}$, b , and the initial value of a ?
- (d) Using these results, describe how an examination of the money-output relationship might be used to measure the credibility of announcements of regime changes.

9.6. Regime changes and the term structure of interest rates. (See Blanchard, 1984; Mankiw and Miron, 1986; and Mankiw, Miron, and Weil, 1987.) Consider an economy where money is neutral. Specifically, assume that $\pi_t = \Delta m_t$ and that r is constant at zero. Suppose that the money supply is given by $\Delta m_t = k\Delta m_{t-1} + \varepsilon_t$, where ε is a white-noise disturbance.

- (a) Assume that the rational-expectations theory of the term structure of interest rates holds (see [9.6]). Specifically, assume that the two-period interest rate is given by $i_t^2 = (i_t^1 + E_t i_{t+1}^1)/2$. i_t^1 denotes the nominal interest rate from t to $t+1$; thus, by the Fisher identity, it equals $r_t + E_t[p_{t+1}] - p_t$.
- (i) What is i_t^1 as a function of Δm_t and k ? (Assume that Δm_t is known at time t .)
- (ii) What is $E_t i_{t+1}^1$ as a function of Δm_t and k ?
- (iii) What is the relation between i_t^2 and i_t^1 ; that is, what is i_t^2 as a function of i_t^1 and k ?
- (iv) How would a change in k affect the relation between i_t^2 and i_t^1 ? Explain intuitively.
- (b) Suppose that the two-period rate includes a time-varying term premium: $i_t^2 = (i_t^1 + E_t i_{t+1}^1)/2 + \theta_t$, where θ is a white-noise disturbance that is independent of ε . Consider the OLS regression $i_{t+1}^1 - i_t^1 = a + b(i_t^2 - i_t^1) + e_{t+1}$.

- (i) Under the rational-expectations theory of the term structure (with $\theta_t = 0$ for all t), what value would one expect for b ? (Hint: for a univariate OLS regression, the coefficient on the right-hand-side variable equals the covariance between the right-hand-side and left-hand-side variables divided by the variance of the right-hand-side variable.)
- (ii) Now suppose that θ has variance σ_θ^2 . What value would one expect for b ?
- (iii) How do changes in k affect your answer to part (ii)? What happens to b as k approaches 1?
- 9.7. (Fischer and Summers, 1989.) Suppose inflation is determined as in Section 9.4. Suppose the government is able to reduce the costs of inflation; that is, suppose it reduces the parameter a in equation (9.9). Is society made better off or worse off by this change? Explain intuitively.
- 9.8. **Solving the dynamic-inconsistency problem through punishment.** (Barro and Gordon, 1983b.) Consider a policymaker whose objection function is $\sum_{t=0}^{\infty} \beta^t (y_t - a\pi_t^e/2)$, where $a > 0$ and $0 < \beta < 1$. y_t is determined by the Lucas supply curve, (9.8), each period. Expected inflation is determined as follows. If π has equaled $\hat{\pi}$ (where $\hat{\pi}$ is a parameter) in all previous periods, then $\pi^e = \hat{\pi}$. If π ever differs from $\hat{\pi}$, then $\pi^e = b/a$ in all subsequent periods.
- (a) What is the equilibrium of the model in all subsequent periods if π ever differs from $\hat{\pi}$?
- (b) Suppose π has always been equal to $\hat{\pi}$, so $\pi^e = \hat{\pi}$. If the monetary authority chooses to depart from $\pi = \hat{\pi}$, what value of π does it choose? What level of its lifetime objective function does it attain under this strategy? If the monetary authority continues to choose $\pi = \hat{\pi}$ every period, what level of its lifetime objective function does it attain?
- (c) For what values of $\hat{\pi}$ does the monetary authority choose $\pi = \hat{\pi}$? Are there values of a , b , and β such that if $\hat{\pi} = 0$, the monetary authority chooses $\pi = 0$?
- 9.9. **Other equilibria in the Barro-Gordon model.** Consider the situation described in Problem 9.8. Find the parameter values (if any) for which each of the following is an equilibrium:
- (a) **One-period punishment.** π_t^e equals $\hat{\pi}$ if $\pi_{t-1} = \pi_{t-1}^e$ and equals b/a otherwise; $\pi = \hat{\pi}$ each period.
- (b) **Severe punishment.** (Abreu, 1988, and Rogoff, 1987.) π_t^e equals $\hat{\pi}$ if $\pi_{t-1} = \pi_{t-1}^e$, equals $\pi_0 > b/a$ if $\pi_{t-1}^e = \hat{\pi}$ and $\pi_{t-1} \neq \hat{\pi}$, and equals b/a otherwise; $\pi = \hat{\pi}$ each period.
- (c) **Repeated discretionary equilibrium.** $\pi = \pi^e = b/a$ each period.
- 9.10. Consider the situation analyzed in Problem 9.8, but assume that there is only some finite number of periods rather than an infinite number. What is the unique equilibrium? (Hint: reason backward from the last period.)
- 9.11. **More on solving the dynamic-inconsistency problem through reputation.** (This is based on Cukierman and Meltzer, 1986.) Consider a policymaker who is in office for two periods and whose objective function is $E[\sum_{t=1}^2 b(\pi_t - \pi_t^e) +$

$c\pi_t - a\pi_t^2/2$. The policymaker is chosen randomly from a pool of possible policymakers with differing tastes. Specifically, c is distributed normally over possible policymakers with mean $\bar{c}$ and variance $\sigma_c^2 > 0$. a and b are the same for all possible policymakers.

The policymaker cannot control inflation perfectly. Instead, $\pi_t = \hat{\pi}_t + \varepsilon_t$, where $\hat{\pi}_t$ is chosen by the policymaker (taking π_t^e as given) and where ε_t is normal with mean zero and variance $\sigma_\varepsilon^2 > 0$. ε_1 , ε_2 , and c are independent. The public does not observe $\hat{\pi}_t$ and ε_t separately, but only π_t . Similarly, the public does not observe c .

Finally, assume that π_2^e is a linear function of π_1 : $\pi_2^e = \alpha + \beta\pi_1$.

- What is the policymaker's choice of $\hat{\pi}_2$? What is the resulting expected value of the policymaker's second-period objective function, $b(\pi_2 - \pi_2^e) + c\pi_2 - a\pi_2^2/2$, as a function of π_2^e ?
- What is the policymaker's choice of $\hat{\pi}_1$ taking α and β as given and accounting for the impact of π_1 on π_2^e ?
- Assuming rational expectations, what is β ? (Hint: use the signal-extraction procedure described in Section 6.3).
- Explain intuitively why the policymaker chooses a lower value of $\hat{\pi}$ in the first period than in the second.

9.12. The tradeoff between low average inflation and flexibility in response to shocks with delegation of control over monetary policy. (Rogoff, 1985.) Suppose that output is given by $y = \bar{y} + b(\pi - \pi^e)$, and that the social welfare function is $\gamma y - a\pi^2/2$, where γ is a random variable with mean $\bar{\gamma}$ and variance σ_γ^2 . π^e is determined before γ is observed; the policymaker, however, chooses π after γ is known. Suppose policy is made by someone whose objective function is $c\gamma y - a\pi^2/2$.

- What is the policymaker's choice of π given π^e , γ , and c ?
- What is π^e ?
- What is the expected value of the true social welfare function, $\gamma y - a\pi^2/2$?
- What value of c maximizes expected social welfare? Interpret your result.

- 9.13.** (a) In the model of reputation analyzed in Section 9.5, is social welfare higher when the policymaker turns out to be a Type 1, or when he or she turns out to be a Type 2?
- (b) In the model of delegation analyzed in Section 9.5, suppose that the policymaker's preferences are believed to be described by (9.23), with $a' > a$, when π^e is determined. Is social welfare higher if these are actually the policymaker's preferences, or if the policymaker's preferences in fact match the social welfare function, (9.9)?

9.14. Money versus interest-rate targeting. (Poole, 1970.) Suppose the economy is described by linear IS and LM curves that are subject to disturbances: $y = c - ai + \varepsilon_{IS}$, $m - p = hy - ki + \varepsilon_{LM}$, where ε_{IS} and ε_{LM} are independent, mean-zero shocks with variances σ_{IS}^2 and σ_{LM}^2 , and where a , h ,

and k are positive. Policymakers want to stabilize output, but they cannot observe y or the shocks, ε_{IS} and ε_{LM} . Assume for simplicity that p is fixed.

- Suppose the policymaker fixes i at some level $\bar{i}$. What is the variance of y ?
- Suppose the policymaker fixes m at some level $\bar{m}$. What is the variance of y ?
- If there are only LM shocks (so $\sigma_{IS}^2 = 0$), does money targeting or interest-rate targeting lead to a lower variance of y ?
- If there are only IS shocks (so $\sigma_{LM}^2 = 0$), does money or interest-rate targeting lead to a lower variance of y ?
- Explain your results in parts (c) and (d) intuitively.
- When there are only IS shocks, is there a policy that produces a variance of y that is lower than either money or interest-rate targeting? If so, what policy minimizes the variance of y ? If not, why not? (Hint: consider the LM curve, $m - p = hy - ki$.)

9.15. Uncertainty and policy. (Brainard, 1967.) Suppose output is given by $y = x + (k + \varepsilon_k)z + u$, where z is some policy instrument controlled by the government and k is the expected value of the multiplier for that instrument. ε_k and u are independent, mean-zero disturbances that are unknown when the policymaker chooses z , and that have variances σ_k^2 and σ_u^2 . Finally, x is a disturbance that is known when z is chosen. The policymaker wants to minimize $E[(y - y^*)^2]$.

- Find $E[(y - y^*)^2]$ as a function of x , k , y^* , σ_k^2 , and σ_u^2 .
- Find the first-order condition for z , and solve for z .
- How, if at all, does σ_u^2 affect how policy should respond to shocks (that is, to the realized value of x)? Thus, how does uncertainty about the state of the economy affect the case for "fine tuning"?
- How, if at all, does σ_k^2 affect how policy should respond to shocks (that is, to the realized value of x)? Thus, how does uncertainty about the effects of policy affect the case for "fine tuning"?

9.16. Growth and seignorage, and an alternative explanation of the inflation-growth relationship. (Friedman, 1971.) Suppose that money demand is given by $\ln(M/P) = a - bi + \ln Y$, and that Y is growing at rate g_Y . What rate of inflation leads to the highest path of seignorage?

9.17. (Cagan, 1956.) Suppose that instead of adjusting their real money holdings gradually toward the desired level, individuals adjust their expectation of inflation gradually toward actual inflation. Thus equations (9.39) and (9.40) are replaced by $m(t) = Ce^{-b\pi^e(t)}$ and $\dot{\pi}^e(t) = \beta[\pi(t) - \pi^e(t)]$, $0 < \beta < 1/b$.

- Follow steps analogous to the derivation of (9.45) to find an expression for $\dot{\pi}^e(t)$ as a function of $\pi(t)$.
- Sketch the resulting phase diagram for the case of $G > S^*$. What are the dynamics of π^e and m ?
- Sketch the phase diagram for the case of $G < S^*$.

Chapter 10

UNEMPLOYMENT

10.1 Introduction: Theories of Unemployment

In almost any economy at almost any time, many individuals appear to be unemployed. That is, there are many people who are not working but who say they want to work in jobs like those held by individuals similar to them, at the wages those individuals are earning.

The possibility of unemployment is a central subject of macroeconomics. There are two basic issues. The first concerns the determinants of average unemployment over extended periods. The central questions here are whether this unemployment represents a genuine failure of markets to clear, and if so, what its causes and consequences are. There is a wide range of possible views. At one extreme is the position that unemployment is largely illusory, or the working out of unimportant frictions in the process of matching up workers and jobs. At the other extreme is the view that unemployment is the result of non-Walrasian features of the economy and that it largely represents a waste of resources.

The second issue concerns the cyclical behavior of the labor market. As described in Chapter 5, the real wage does not appear to be highly procyclical. This is consistent with the view that the labor market is Walrasian only if labor supply is highly elastic or if shifts in labor supply play an important role in employment fluctuations. But as we saw in Section 4.10, there is little support for the hypothesis of highly elastic labor supply. And it seems unlikely that shifts in labor supply are central to fluctuations. The remaining possibility is that the labor market is not Walrasian, and that its non-Walrasian features are central to its cyclical behavior. That possibility is the focus of this chapter.

The issue of why shifts in labor demand appear to lead to large movements in employment and only small movements in the real wage is important to all theories of fluctuations. For example, we saw in Chapter 6 that if the real wage is highly procyclical in response to demand shocks, it is essentially impossible for the small barriers to nominal adjustment to generate substantial nominal rigidity. In the face of a decline in aggregate

not reflect ability differences. Second, higher-wage industries have higher capital-labor ratios, which suggests that they need more skilled workers. Third, workers in higher-wage industries have higher measured ability (in terms of education, experience, and so on); thus it seems likely that they have higher unmeasured ability. Finally, the same patterns of interindustry earnings differences occur, although less strongly, among self-employed workers.

The hypothesis that estimated interindustry wage differences reflect unmeasured ability cannot easily account for all of the findings about these differences, however. First, quantitative attempts to estimate how much of the differences can plausibly be due to unmeasured ability generally leave a substantial portion of the differences unaccounted for (see, for example, Katz and Summers, 1989). Second, the unmeasured-ability hypothesis cannot readily explain Gibbons and Katz's findings about the wage cuts of displaced workers. Third, the estimated wage premia are higher in industries where profits are higher; this is not what the unmeasured-ability hypothesis naturally predicts. Finally, industries that pay higher wages generally do so in all occupations, from janitors to managers; it is not clear that unmeasured ability differences should be so strongly related across occupations. Thus, although the view that interindustry wage differences reflect unmeasured ability is troubling for rent-based explanations of those differences, it does not definitively refute them.

The second aspect of this literature's findings that is not easily accounted for by efficiency-wage theories concerns the characteristics of industries that pay high wages. As described above, higher-wage industries tend to have higher capital-labor ratios, more educated and experienced workers, and higher profits. In addition, they have larger establishments, and larger fractions of male and of unionized workers (Dickens and Katz, 1987b). No single efficiency-wage theory predicts all of these patterns. As a result, authors who believe that the estimated interindustry wage differences reflect rents tend to resort to complicated explanations of them. Dickens and Katz and Krueger and Summers, for example, appeal to a combination of efficiency-wage theories based on imperfect monitoring, efficiency-wage theories based on workers' perceptions of fairness, and worker power in wage determination.

In sum, the literature on interindustry wage differences has identified an interesting set of regularities that differ greatly from what simple theories of the labor market predict. The reasons for those regularities, however, have not yet been convincingly identified.

Problems

- 10.1. **Union wage premia and efficiency wages.** (Summers, 1988.) Consider the efficiency-wage model analyzed in equations (10.12)-(10.19). Suppose, however, that fraction f of workers belong to unions that are able to obtain

a wage that exceeds the nonunion wage by proportion μ . Thus, $w_u = (1 + \mu)w_n$, where w_u and w_n denote wages in the union and nonunion sectors; and the average wage, w_a , is given by $f w_u + (1 - f)w_n$. Nonunion employers continue to set their wages freely; thus (by the same reasoning used to derive [10.15] in the text), $w_n = (1 - bu)w_a / (1 - \beta)$.

(a) Find the equilibrium unemployment rate in terms of β , b , f , and μ .

(b) Suppose $\mu = f = 0.15$.

(i) What is the equilibrium unemployment rate if $\beta = 0.06$ and $b = 1$? By what proportion is the cost of effective labor higher in the union sector than in the nonunion sector?

(ii) Repeat part (i) for the case of $\beta = 0.03$ and $b = 0.5$.

10.2. Efficiency wages and bargaining. Summers (1988, p. 386) states "In an efficiency wage environment, firms that are forced to pay their workers premium wages suffer only second-order losses. In almost any plausible bargaining framework, this makes it easier for workers to extract concessions." This problem asks you to investigate this claim.

Consider a firm with profits given by $\pi = (eL)^\alpha / \alpha - wL$, $0 < \alpha < 1$, and a union with objective function $U = w - x$, where x is an index of its workers' outside opportunities. Assume that the firm and the union bargain over the wage, and that the firm then chooses L taking w as given.

(a) Suppose that $e \equiv 1$, so that efficiency-wage considerations are absent.

(i) What value of L does the firm choose given w ? What is the resulting level of profits?

(ii) Suppose that the firm and the union choose w to maximize $U^\gamma \pi^{1-\gamma}$, where $0 < \gamma < \alpha$ indexes the union's power in the bargaining (this is known as the *Nash bargaining solution*). What level of w do they choose?

(iii) What is $\partial(\ln w) / \partial \gamma$ at $\gamma = 0$?

(b) Suppose that e is given by equation (10.12) in the text: $e = [(w - x)/x]^\beta$ for $w > x$, where $0 < \beta < 1$.

(i) What value of L does the firm choose given w ? What is the resulting level of profits?

(ii) Suppose that the firm and the union choose w to maximize $U^\gamma \pi^{1-\gamma}$, $0 < \gamma < \alpha$. What level of w do they choose? (Hint: for the case of $\beta = 0$, your answer should simplify to your answer in part [a][ii].)

(iii) What is $\partial(\ln w) / \partial \gamma$ at $\gamma = 0$? Is this elasticity higher with efficiency wages than without, as Summers argues? Is the impact of bargaining on the wage qualitatively different with efficiency wages, as Summers implies?

10.3 Describe how each of the following affect equilibrium employment and the wage in the Shapiro-Stiglitz model:

(a) An increase in workers' discount rate, ρ .

- (b) An increase in the job breakup rate, b .
- (c) A positive multiplicative shock to the production function (that is, suppose the production function is $AF(L)$, and consider an increase in A).
- (d) An increase in the size of the labor force, $\bar{L}$.

10.4 Suppose that in the Shapiro-Stiglitz model, unemployed workers are hired according to how long they have been unemployed rather than at random; specifically, suppose that workers who have been unemployed the longest are hired first.

- (a) Consider a steady state where there is no shirking. Derive an expression for how long it takes a worker who becomes unemployed to get a job as a function of b, L, N , and $\bar{L}$.
- (b) Let V_U be the value of being a worker who is newly unemployed. Derive an expression for V_U as a function of the time it takes to get a job, workers' discount rate (ρ), and the value of being employed (V_E).
- (c) Using your answers to parts (a) and (b), find the no-shirking condition for this version of the model.
- (d) How, if at all, does the assumption that the longer-term unemployed get priority affect the equilibrium unemployment rate?

10.5 The fair wage-effort hypothesis. (Akerlof and Yellen, 1990.) Suppose there is a large number of firms, N , each with profits given by $F(eL) - wL$, $F'(\bullet) > 0$, $F''(\bullet) < 0$. L is the number of workers the firm hires, w is the wage it pays, and e is workers' effort. Effort is given by $e = \min[w/w^*, 1]$, where w^* is the "fair wage"; that is, if workers are paid less than the fair wage, they reduce their effort in proportion to the shortfall. Assume that there are $\bar{L}$ workers who are willing to work at any positive wage.

- (a) If a firm can hire workers at any wage, what value (or range of values) of w yields the highest profits? For the remainder of the problem, assume that if the firm is indifferent over a range of possible wages, it pays the highest value in this range.
- (b) Suppose w^* is given by $w^* = \bar{w} + a - bu$, $b > 0$, where u is the unemployment rate and $\bar{w}$ is the average wage paid by the firms in this economy.
 - (i) Given your answer to part (a), what wage does the representative firm pay if it can choose w freely (taking $\bar{w}$ and u as given)?
 - (ii) Under what conditions does the equilibrium involve positive unemployment and no constraints on firms' choice of w ? (Hint: in this case, equilibrium requires that the representative firm, taking $\bar{w}$ as given, wishes to pay $\bar{w}$.) What is the unemployment rate in this case?
 - (iii) Under what conditions is there full employment?
- (c) Suppose the representative firm's production function is modified to be $F(AL_1 + L_2)$, $A > 1$, where L_1 and L_2 are the numbers of high-

productivity and low-productivity workers the firm hires. Assume that the fair wage for type- i workers is given by $w_i^* = (\bar{w}_1 + \bar{w}_2)/2 - bu_i$, where $\bar{w}_i$ is the average wage paid to workers of type i and u_i is their unemployment rate. Finally, assume there are $\bar{L}$ workers of each type.

- (i) In equilibrium, is there unemployment among high-productivity workers? Explain. (Hint: if u_1 is positive, firms are unconstrained in their choice of w_1 .)
- (ii) In equilibrium, is there unemployment among low-productivity workers? Explain.

10.6 Implicit contracts without variable hours. Suppose that each worker must either work a fixed number of hours or be unemployed. Let C_i^E denote the consumption of employed workers in state i , and C_i^U the consumption of unemployed workers. The firm's profits in state i are therefore $A_i F(L_i) - [C_i^E L_i + C_i^U (\bar{L} - L_i)]$, where $\bar{L}$ is the number of workers. Similarly, workers' expected utility in state i is $(L_i/\bar{L})[U(C_i^E) - K] + [(\bar{L} - L_i)/\bar{L}]U(C_i^U)$, where $K > 0$ is the disutility of working.

- (a) Set up the Lagrangian for the firm's problem of choosing the L_i 's, C_i^E 's, and C_i^U 's to maximize expected profits subject to the constraint that the representative worker's expected utility is u_0 .³³
- (b) Find the first-order conditions for L_i , C_i^E , and C_i^U . How, if at all, do C_i^E and C_i^U depend on the state? What is the relation between C_i^E and C_i^U ?
- (c) After A is realized and some workers are chosen to work and others are chosen to be unemployed, which workers are better off?

10.7 Unemployment insurance. (This follows Feldstein, 1976.) Consider a firm with revenues $AF(L)$. A has two possible values, A_B and A_G ($A_B < A_G$), each of which occurs half the time. Workers who are employed when $A = A_G$ and unemployed when $A = A_B$ receive an unemployment insurance benefit of $B > 0$ when $A = A_B$. Workers are risk-neutral; thus the representative worker's expected utility is $U = (w - K)/2 + \{(L_B/L_G)(w - K) + [(L_G - L_B)/L_G]B\}/2$, where w is the wage (which is assumed without loss of generality to be independent of the state), K is the disutility of working, and L_B and L_G are employment in the two states. The firm's expected profits are $[A_G F(L_G) - wL_G]/2 + [A_B F(L_B) - wL_B - fB(L_G - L_B)]/2$, where f is the fraction of unemployment benefits that are paid by the firm. Assume $0 \leq f \leq 1$.³⁴

- (a) Set up the Lagrangian for the firm's problem of choosing w , L_G , and L_B to maximize expected profits subject to the constraint that workers' expected utility is u_0 .
- (b) Find the first-order conditions for w , L_G , and L_B .

³³For simplicity, neglect the constraint that L cannot exceed $\bar{L}$. Accounting for this constraint, one would find that for A_i above some critical level, L_i would equal $\bar{L}$ rather than being determined by the condition derived in part (b), below.

³⁴In the United States, a firm's unemployment insurance taxes only partly account for the extent to which the firm's workers obtain unemployment insurance; that is, the taxes are only partially *experience-rated*. Thus f is between zero and one.

(c) Show that a fall in f (or a rise in B if $f < 1$) reduces L_B .

(d) Show that a fall in f (or a rise in B if $f < 1$) raises L_G .

10.8 Implicit contracts under asymmetric information. (Azariadis and Stiglitz, 1983.) Consider the model of Section 10.5. Suppose, however, that only the firm observes A . In addition, suppose there are only two possible values of A , A_B and A_G ($A_B < A_G$), each occurring with probability one-half.

We can think of the contract as specifying w and L as functions of the firm's announcement of the state, and as being subject to the restriction that it is never in the firm's interest to announce a state other than the actual one; formally, the contract must be *incentive-compatible*.

(a) Is the efficient contract under symmetric information derived in Section 10.5 incentive-compatible under asymmetric information? Specifically, if A is A_B , is the firm better off claiming that A is A_G (so that C and L are given by C_G and L_G) rather than that it is A_B ? And if A is A_G , is the firm better off claiming it is A_B rather than A_G ?

(b) One can show that the constraint that the firm not prefer to claim that the state is bad when it is good is not binding, but that the constraint that it not prefer to claim that the state is good when it is bad is binding. Set up the Lagrangian for the firm's problem of choosing C_G, C_B, L_G , and L_B subject to the constraints that workers' expected utility is u_0 and that the firm is indifferent about which state to announce when A is A_B . Find the first-order conditions for C_G, C_B, L_G , and L_B .

(c) Show that the marginal product and the marginal disutility of labor are equated in the bad state—that is, that $A_B F'(L_B) = V'(L_B)/U'(C_B)$.

(d) Show that there is "overemployment" in the good state—that is, that $A_G F'(L_G) < V'(L_G)/U'(C_G)$.

(e) Is this model helpful in understanding the high level of average unemployment? Is it helpful in understanding the large size of employment fluctuations?

10.9 Does worker influence on the wage after shocks to labor demand are realized affect the cyclical characteristics of the labor market?

(a) (This follows McDonald and Solow, 1981.) Consider a union with the objective function $[U(w) - K]L + U(w_u)(N - L)$, $U'(\cdot) > 0$, where N is the number of union members, L is the number who are employed, $K > 0$ is the disutility of working, w is the wage, and w_u is unemployment compensation. The firm's profits are $AL^\alpha/\alpha - wL$, $A > 0$, $0 < \alpha < 1$. The union sets w after A is known, and the firm then chooses L given w and A . (Assume throughout the problem that the constraint that L cannot exceed N is not binding.)

(i) What is the firm's choice of L given w and A ?

(ii) Given its knowledge of how the firm will behave, what is the union's choice of w given A ? Given this, how does L vary with A ?

(b) Given the union's objective function, what is labor supply under spot markets—that is, if the union takes w as given and chooses L to

maximize its objective function? How do w and L vary with A under spot markets?

- (c) Suppose the union's objective function is $wL - [\sigma/(\sigma + 1)]L^{(\sigma+1)/\sigma}$, $\sigma > 0$, instead of the expression in part (a).

(i) How do w and L vary with A under spot markets?

(ii) Redo part (a)(ii) using the modified union objective function. Does assuming that the wage is determined by the union rather than by spot markets affect the elasticities of w and L with respect to A ?

10.10 Does worker influence on the wage and employment after shocks to labor demand are realized affect the cyclical characteristics of the labor market?

- (a) (This is based on McDonald and Solow, 1981.) Consider a union and a firm with the objective functions assumed in part (a) of Problem 10.9. The union chooses w and L given A , subject to the constraint that the firm's profits must be at least some level, π_0 .

(i) Set up the Lagrangian for the union's maximization problem.

(ii) Find the first-order conditions for w and L .

(iii) What role does w play in this model?

(iv) How does L vary with A ? Compare your answer with your finding in part (b) of Problem 10.9 concerning how L varies with A under spot markets.

- (b) Suppose the union's objective function is given instead by the expression in part (c) of Problem 10.9. Redo parts (i) and (ii) of part (a) using the modified union objective function. Compare how L varies with A with your finding in part (c)(i) of Problem 10.9 concerning how it varies under spot markets.

10.11 The Harris-Todaro model. (Harris and Todaro, 1970.) Suppose there are two sectors. Jobs in the primary sector pay w_p ; jobs in the secondary sector pay w_s . Each worker decides which sector to be in. All workers who choose the secondary sector obtain a job. But there is a fixed number, N_p , of primary-sector jobs. These jobs are allocated at random among workers who choose the primary sector. Primary-sector workers who do not get a job are unemployed, and receive an unemployment benefit of b . Workers are risk-neutral, and there is no disutility of working. Thus the expected utility of a primary-sector worker is $qw_p + (1 - q)b$, where q is the probability of a primary-sector worker getting a job.

- (a) What is equilibrium unemployment as a function of w_p , w_s , N_p , b , and the size of the labor force, $\bar{N}$?

(b) How does an increase in N_p affect unemployment? Explain intuitively why, even though unemployment takes the form of workers waiting for primary-sector jobs, increasing the number of these jobs can increase unemployment.

- (c) What are the effects of an increase in the level of unemployment benefits?

10.12 Partial-equilibrium search. Consider a worker searching for a job. Wages, w , have a probability density function across jobs $f(w)$ that is known to the worker; let $F(w)$ be the associated cumulative distribution function. Each time the worker samples a job from this distribution, he or she incurs a cost of C , where $0 < C < E[w]$. When the worker samples a job, he or she can either accept it (in which case the process ends) or sample another job. The worker maximizes the expected value of $w - nC$, where w is the wage paid in the job the worker eventually accepts, and n is the number of jobs the worker ends up sampling.

Let V denote the expected value of $w - n'C$ of a worker who has just rejected a job, where n' is the number of jobs the worker will sample from that point on.

- (a) Explain why the worker accepts a job offering $\hat{w}$ if $\hat{w} > V$, and rejects it if $\hat{w} < V$. (A search problem where the worker accepts a job if and only if it pays above some cutoff level is said to exhibit the *reservation-wage property*.)
- (b) Explain why V satisfies $V = F(V)V + \int_{w=V}^{\infty} wf(w)dw - C$.
- (c) Show that an increase in C reduces V .
- (d) In this model, does a searcher ever want to accept a job that he or she has previously rejected?

10.13 In the setup described in Problem 10.12, suppose that w is distributed uniformly on $[\mu - a, \mu + a]$, and that $C < \mu$.

- (a) Find V in terms of μ , a , and C .
- (b) How does an increase in a affect V ? Explain intuitively.

10.14 Describe how each of the following affects equilibrium employment in the model of Section 10.8:

- (a) An increase in the job breakup rate, b .
- (b) An increase in the interest rate, r .
- (c) An increase in the effectiveness of matching, K .

10.15 Suppose we replace the assumption in equation (10.76) that the worker and the firm divide the surplus from their relationship equally with the assumption that fraction f of the surplus goes to the worker and fraction $1 - f$ goes to the firm: $(1 - f)(V_E - V_U) = f(V_F - V_U)$.

- (a) How does this change in the model affect the equation implicitly defining E , (10.85)?
- (b) How does a change in f affect the equilibrium level of E ?

10.16 Consider the model of Section 10.8. Suppose the economy is initially in equilibrium, and that A then falls permanently. Suppose, however, that entry and exit are ruled out; thus the total number of jobs, $F + V$, remains constant. How do unemployment and vacancies behave over time in response to the fall in A ?

10.17 The efficiency of the decentralized equilibrium in a search economy. Consider the model of Section 10.8. Let the interest rate, r , approach zero, and assume that the firms are owned by the households; thus welfare can be measured as the sum of utility and profits per unit time, which equals $AE - (F + V)C$. Letting N denote the total number of jobs, we can therefore write welfare as $W(N) = AE(N) - NC$, where $E(N)$ gives equilibrium employment as a function of N .

- (a) Use the matching function, (10.68), and the steady-state condition, (10.69), to derive an expression for the impact of a change in the number of jobs on employment, $E'(N)$, in terms of $N, \bar{L}, E(N), \gamma$, and β .
- (b) Substitute your result in part (a) into the expression for $W(N)$ to find $W'(N)$ in terms of $N, \bar{L}, E(N), \gamma, \beta$, and A .
- (c) Use (10.82) and the facts that $a = bE/(\bar{L} - E)$ and $\alpha = bE/V$ to find an expression for C in terms of $N_{EQ}, \bar{L}, E(N_{EQ})$, and A , where N_{EQ} is the number of jobs in the decentralized equilibrium.
- (d) Use your results in parts (b) and (c) to show that if $\beta + \gamma = 1$, $W'(N_{EQ}) > 0$ if $\gamma > \frac{1}{2}$ and $W'(N_{EQ}) < 0$ if $\gamma < \frac{1}{2}$.
- (e) If γ is $\frac{1}{2}$ but $\beta + \gamma$ is not necessarily 1, what determines the sign of $W'(N_{EQ})$?